GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

SPONSORED PROJECT INITIATION

Date: April 25, 1980

Project Title: Studies on Controllable Motions

Project No: E-23-651

Project Director: Dr. Andrew W. Marris

Sponsor: National Science Foundation

Agreement Period: From April 1, 1980 Until September 30, 1982 (Grant Peric

Type Agreement: Grant No. CME-7820240, dated April 10, 1980

Amount:
$61,828 NSF (E-23-651)
4,782 GIT (E-23-343)
$66,610 TOTAL

Reports Required: Annual Progress Report(s); Final Project Report

Sponsor Contact Person (s):

Technical Matters

NSF PROGRAM OFFICIAL

George K. Lea
Program Director for Fluid Mechanics
Mechanical Sciences and Engineering Group
Division of Civil and Mechanical Engineering
Directorate for Engineering and Applied Science
National Science Foundation
Washington, D. C. 20550
(202) 632-5787

Contractual Matters
(thru OCA)

NSF GRANTS OFFICIAL

Karen L. Sandberg
AAEO/EAS Branch, Section I
Division of Grants and Contracts
Directorate for Administration
National Science Foundation
Washington, D. C. 20550
(202) 632-5892

Defense Priority Rating: N/A

Assigned to: Engineering Science & Mechanics (School/~~Laboratory~~)

COPIES TO:

Project Director
Division Chief (EES)
School/Laboratory Director
Dean/Director—EES
Accounting Office
Procurement Office
Security Coordinator (OCA)
Reports Coordinator (OCA)

Library, Technical Reports Section
EES Information Office
EES Reports & Procedures
Project File (OCA)
Project Code (GTRI)
Other C. E. Smith

GEORGIA INSTITUTE OF TECHNOLOGY OFFICE OF CONTRACT ADMINISTRATION

SPONSORED PROJECT TERMINATION SHEET

Date 11-12-82

Project Title: Studies on Controllable Motions

Project No: E-23-651

Project Director: Dr. Andrew W. Marris

Sponsor: NSF

Effective Termination Date: 9/30/82

Clearance of Accounting Charges: 9/30/82

Grant/Contract Closeout Actions Remaining:

- [ ] Final Invoice and Closing Documents
- [x] Final Fiscal ~~Report~~ Accounting (FCTR)
- [x] Final Report of Inventions
- [ ] Govt. Property Inventory & Related Certificate
- [ ] Classified Material Certificate
- [ ] Other ______________

Assigned to: ESM (School/~~Laboratory~~)

COPIES TO:

Administrative Coordinator
Research Property Management
Accounting
Procurement/EES Supply Services

Research Security Services
Reports Coordinator (OCA)
Legal Services (OCA)
Library

EES Public Relations (2)
Computer Input
Project File
Other ______________

ENGINEERING COLLEGE
GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL OF ENGINEERING SCIENCE
AND MECHANICS

225 NORTH AVENUE, N.W.
ATLANTA, GEORGIA 30332

May 15, 1981

Dr. George K. Lea, Program Director
for Fluid Mechanics
Mechanical Sciences and Engineering Group
Division of Civil and Mechanical Engineering
Directorate for Engineering
and Applied Sciences
National Science Foundation
Washington, D. C. 20550

Re: NSF Grant No. CME 7820240

Dear Dr. Lea:

This letter is my progress report for the period 4/1/80 to 3/31/81 for my NSF Grant No. CME 7820240 entitled "Studies on Controllable Motions".

I was first concerned with the determination and study of universal motions of simple fluids.

At the beginning of the grant period I had solved completely the problem of determining the steady, rotational, universal motions of incompressible simple materials in the case when the proper numbers of the first Rivlin-Ericksen tensor are not all constant. This work is contained in my paper "Steady Universal Motions of Rivlin Ericksen Fluids" Archive for Rational Mechanics and Analysis 69, 335-380 (1979), and the paper in collaboration with C.-C. Wang: "Proof that Motions Obtained in the Preceding Paper by Marris are Universal for all Incompressible Isotropic Simple Materials", Archive for Rational Mechanics and Analysis 69, 381-390, (1979).

The problem remaining was to delimit the possible motions in the case when the proper numbers of the Rivlin-Ericksen tensor are constant.

The approach to this problem, as in the one already solved, was to use a result of Fosdick and Carroll (see references in the second of above papers) that the universal motions are obtained directly from the universal deformations of finite elasticity, by making the constants of the elasticity problem such functions of time as to make the acceleration lamellar. Thus in order to settle the fluid mechanics problem it was necessary first to settle the elasticity problem. They are essentially the same problem. I enclose with this letter a manuscript containing my discoveries concerning this problem. This manuscript is currently being reviewed for publication.

I summarize the results as follows:

1) We indicate sixteen globally independent algebraic equations in sixteen variables. If these equations can be solved for the sixteen variables, showing that the variables are all constant, then there are no more solutions to the elasticity problem or the fluid mechanics problem. To clinch the problem completely one must use a computer program geared to do algebraic eliminations, for example the MACSYMA program.

2) If there is a new solution then the curvatures of the vector-lines of the proper vectors must satisfy at least one algebraic relation.

3) I showed that, if the proper numbers are functionally related, this relation must be of one particular form, otherwise there are no new solutions.

4) The analysis separates kinematic results from kinetic results, and the purely kinematic part forms a basis for the study of deformation of anisotropic materials under constant hydrostatic pressure.

This work was presented in an invited lecture in the continuum mechanics session of the 17th Midwestern Mechanics Conference, University of Michigan, Ann Arbor, Michigan, May 6-8, 1981.

Next, in accordance with the proposal, I was concerned with universal motions of homogeneous incompressible linearly viscous fluids. In particular I have been studying both steady and unsteady plane flows. This work is contained in the paper "Remarks on Plane Universal Navier Stokes Motions", which is currently in press for the Archive for Rational Mechanics and Analysis. In this research steady universal plane motions are delimited completely. Also in this paper it is shown that for any unsteady plane motion possessing an acceleration potential, that is a motion appropriate to a homogeneous incompressible inviscid fluid, the (unsteady) vector-lines of curl $\underset{\sim}{\omega}$, where $\underset{\sim}{\omega}$ is the vorticity, are the curves of constant vorticity magnitude, and moreover these curves always contain the same fluid particles. It seems possible that this result relates to the fact that certain highly unsteady plane motions, (for example the turbulent Karman vortex street), tend to maintain their original laminar form.

A the present time I am continuing the study of unsteady plane flows.

Yours sincerely,

A. W. Marris
Regents' Professor

ew
Enclosure

UNIVERSAL DEFORMATIONS IN FINITE ELASTICITY

A. W. MARRIS

School of Engineering Science and Mechanics
Georgia Institute of Technology
Atlanta, Georgia 30332

February 1981

TABLE OF CONTENTS

INTRODUCTION

Ericksen (1954 [1]) attacked the general problem of determining the deformations that can be produced in every isotropic incompressible perfectly elastic body by the application of surface tractions when body forces are absent. Two cases proved intractable at that time, a case when the deformation tensor $\underset{\sim}{c}$ had equal proper numbers, and the case when its proper numbers were all constant.* Marris and Shiau (1970 [1]) showed there were no further solutions in the first category. The problem of constant proper numbers has remained unsolved.

We give a brief history of the researches on the constant proper numbers case. Fosdick (1966 [1]) noted that the known deformation

$$r = aR, \quad \theta = b\Theta, \quad z = cZ, \quad a^2bc = 1, \quad b \neq 1$$

represented a solution for the case of constant proper numbers. Singh and Pipkin (1965 [1]) gave the new solution

$$r = aR, \quad \theta = b \log R + c\Theta, \quad z = dZ, \quad a^2cd = 1 \qquad (I.1)$$

and they noted that a special case of this deformation corresponding to $a^2c = 1$, $b^2 + c^2 = 1$, had been found by Klingbeil and Shield (1966 [2]). Fosdick and Schuler (1969 [1]) characterized all the universal deformations which were plane deformations (with uniform transverse stretch), and showed that, beyond homogeneous deformations, the above solution was the only plane deformation in the class of constant proper numbers. Fosdick (1971 [1]) showed that there were

*A part of Ericksen's analysis was based on a deeply seated geometrical theorem originally given without adequate proof by Hamel. This theorem was proved by the writer in (1973 [1]).

no new solutions for the class of radially symmetric deformations. Kafader (1972 [1]) proved that (I.1) is the only possible solution in the case when any two of $\underset{\sim}{c}$'s proper numbers are equal.* For the remaining case of distinct and constant proper numbers, Kafader also proved that if the abnormality of the vector-field of any one of $\underset{\sim}{c}$'s proper vectors vanishes, then no new solutions exist. Finally, the writer showed in (1975 [1]) that no new solutions exist when any two of the abnormalities of the proper vectors of $\underset{\sim}{c}$ are constant.

It is seen that all the past work has been directed to showing that no new solutions exist for special classes of deformations. No attempt has been made to attack the problem from the other end, that is, to prove that if a new deformation exist it must be of a certain type. In the present work we attempt this.

We prove the following theorems:*

MAIN THEOREM I.1.

If there exist a new class of solutions then it must be such that the curvatures and abnormalities of the fields of the unit proper vectors and the proper numbers are functionally related, thus

$$F\left(\pi_{aa}, \pi_{ab}, \frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}, \frac{\sigma_3^2}{\sigma_1^2}\right) = 0 \qquad \text{(I.2)}$$

*The proper numbers π_a and abnormalities π_{aa} and curvatures π_{ab} referred to in (I.2), (I.3) and (I.4) are defined in Chapter 1.

where F is a symmetric polynomial in its arguments, and is homogeneous in the nine abnormalities and curvatures π_{aa}, π_{ab}. *The form of F is such that it is invariant when the basis of proper vectors is transformed by reflection from a right-handed to a left-handed system, and for a ninety degree rotation about the direction determined by a proper vector.*

MAIN THEOREM I.2

If the condition (I.2) is satisfied by a functional relation ϕ_n *among the proper numbers only, so that*

$$F = \phi_n \left(\frac{\sigma_1^2}{\sigma_2^2} , \frac{\sigma_2^2}{\sigma_3^2} \right) \quad G \left(\pi_{aa}, \pi_{ab}, \frac{\sigma_1^2}{\sigma_2^2} , \frac{\sigma_2^2}{\sigma_3^2} \right) = 0 , \qquad \text{(I.3)}$$

then ϕ_n *must be of the form*

$$\phi_n = \frac{\left(\frac{\sigma_3^2}{\sigma_1^2} - 1 \right)}{\left(\frac{\sigma_2^2}{\sigma_1^2} - 1 \right)} - k_n \frac{\sigma_3^2}{\sigma_2^2} , \qquad \text{(I.4)}$$

where k_n *is a constant which can have a discrete number of values. If there exist a new solution in which the proper numbers are functionally related, this relation must be of the form**

$$\phi_n = 0 . \qquad \text{(I.5)}$$

**The known solution (I.1) corresponds to* $k_n = 0$.

The difficulty with this problem lies not so much in generating polynomial integrals as necessary conditions, but rather in proving that these conditions are completely independent. These integrals must involve a minimum of ten variables so that even if they turn out to be of relatively low degree they will be extremely long expressions.* While such expressions may appear to be globally independent, one is always faced with the possibility that common factors may occur at late stages of the elimination. The problem appears to be well suited to the new computer symbolic mathematics systems (1979 [1]).

We give ten polynomial integrals effectively involving sixteen arguments. These are the ten variables indicated above and the six variables u_1, u_2, u_3, $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$, defined in Chapter 1. Complicated as they are, we are confident of their correctness, because they allow several independent checks**. Our Theorem 1.1 is derived by eliminating the six latter variables from these conditions.*** The conditions are derived in sets of three, it appears to be possible to derive an infinite number of conditions of increasingly high order by taking the gradients of these integrals.

---

*i.e., eight ratios of the type $\frac{\pi_{ab}}{\pi_{11}}$ and the two independent proper numbers.

**They satisfy the invariance conditions introduced in Chapter 1. Also badly directed manipulations of these conditions lead to identities, as discussed in the sub-section entitled Parenthesis.

***This shows that the conditions in question enjoy a degree of independence.

We indicate six further integrals which appear to be independent of the first ten. If indeed they are so, then elimination would show that the ratios of the abnormalities are constant. We are forced to leave the verification of this to the computer.

In the Appendix we give a proof that no new solutions exist in the special case when the ratios of the abnormalities are constant. To complete the proof that no new solutions exist we require a computer symbolic program to deliver the six indicated integrals in complete form,* and then to show that the elimination is possible.

In presenting this work we consider it important at the beginning to separate those conditions which are of geometrical origin, from those arising from the equilibrium conditons. For one thing, solutions for the geometrical problem may be important in the study of deformations of aeolotropic materials. Also, while at first sight the equilibrium conditions appear to give two vector equations, it turns out that only one of these is independent of the geometrical conditions.

Chapters 1 and 2 deal with the geometrical conditions; the equilibrium conditions are introduced in Chapter 3. The ten basic integrals are developed in Chapters 4, 5 and 6. In a short section entitled Parenthesis we indicate various combinations that lead to identities. Summarizing our futile attempts to obtain any further integrals of the same order as the original ten, it recounts the various checks that have been made on the basic conditions. In Chapter 7 we

*The two sets of three are obtained from each other by the ninety degree rotation of axes.

prove the first main theorem by performing the elimination after appropriate numerical substitutions. This simplified version indicates the scope of the necessary general elimination process. In Chapter 8 we prove the second main theorem. In Chapter 9 we indicate the additional integrals necessary to complete the proof. The Appendix deals with the special case when the ratios of the abnormalities are constant.

1. GEOMETRY OF ISOCHORIC DEFORMATIONS WITH CONSTANT PROPER NUMBERS.

PRELIMINARIES.

A deformation is defined by the invertible mapping $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X})$. Associated with the deformation are the deformation gradients $\underset{\sim}{F}$ and $\underset{\sim}{f}$ given by the double tensors*

$$\underset{\sim}{F} = \frac{d\underset{\sim}{x}}{d\underset{\sim}{X}} = \sigma_1 \underset{\sim}{e}_1 \underset{\sim}{E}_1 + \sigma_2 \underset{\sim}{e}_2 \underset{\sim}{E}_2 + \sigma_3 \underset{\sim}{e}_3 \underset{\sim}{E}_3 \, , \tag{1.1}$$

$$\underset{\sim}{f} = \frac{d\underset{\sim}{X}}{d\underset{\sim}{x}} = \frac{1}{\sigma_1} \underset{\sim}{E}_1 \underset{\sim}{e}_1 + \frac{1}{\sigma_2} \underset{\sim}{E}_2 \underset{\sim}{e}_2 + \frac{1}{\sigma_3} \underset{\sim}{E}_3 \underset{\sim}{e}_3 \, , \tag{1.2}$$

and the right and left Cauchy-Green tensors are $\underset{\sim}{C}$ and $\underset{\sim}{c}^{-1}$ where

$$\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F} = \sigma_1^2 \underset{\sim}{E}_1 \underset{\sim}{E}_1 + \sigma_2^2 \underset{\sim}{E}_2 \underset{\sim}{E}_2 + \sigma_3^2 \underset{\sim}{E}_3 \underset{\sim}{E}_3 \, , \tag{1.3}$$

and

$$\underset{\sim}{c} = f^T f = \frac{1}{\sigma_1^2} \underset{\sim}{e}_1 \underset{\sim}{e}_1 + \frac{1}{\sigma_2^2} \underset{\sim}{e}_2 \underset{\sim}{e}_2 + \frac{1}{\sigma_3^2} \underset{\sim}{e}_3 \underset{\sim}{e}_3 \, . \tag{1.4}$$

The ortho-normal bases $\underset{\sim}{E}_\Lambda$ and $\underset{\sim}{e}_a$ point along the proper vectors of $\underset{\sim}{C}$ and $\underset{\sim}{c}$, and σ_1^2, σ_2^2, σ_3^2 and their reciprocals are the corresponding proper numbers. They are considered to be constant and distinct. For an isochoric deformation we require that

$$\sigma_1 \, \sigma_2 \, \sigma_3 = 1 \, . \tag{1.5}$$

*The symbolism $\underset{\sim}{F} = \frac{d\underset{\sim}{x}}{d\underset{\sim}{X}}$ indicates that $d\underset{\sim}{x} = \underset{\sim}{F} d\underset{\sim}{X}$

The gradient of the vector field $\underset{\sim}{e}_c(\underset{\sim}{x})$ referred to the basis $\underset{\sim}{e}_a$ is written

$$\text{grad } \underset{\sim}{e}_c = \gamma^{b}_{ac}\, \underset{\sim}{e}_a \underset{\sim}{e}_b \;,\quad \gamma^{c}_{ac} = 0,\quad \gamma^{b}_{ac} = -\gamma^{c}_{ab} \;. \tag{1.6}$$

The gradient of $\underset{\sim}{E}_c(\underset{\sim}{X})$ is given similarly by

$$\text{GRAD } \underset{\sim}{E}_C = \Gamma^{B}_{AC}\, \underset{\sim}{e}_A \underset{\sim}{e}_B \;,\quad \Gamma^{C}_{AC} = 0 \;,\quad \Gamma^{B}_{AC} = -\Gamma^{C}_{AB} \;. \tag{1.7}$$

Rather than the gammas, we choose as the main vehicle for our analysis the functions π_{ab}, which are the components of curl $\underset{\sim}{e}_a$, thus

$$\pi_{ab} = \underset{\sim}{e}_b \cdot \text{curl } \underset{\sim}{e}_a \;,$$

so that

$$\text{curl } \underset{\sim}{e}_1 = \pi_{11}\, \underset{\sim}{e}_1 + \pi_{12}\, \underset{\sim}{e}_2 + \pi_{13}\, \underset{\sim}{e}_3 \;, \; etc.^{*} \tag{1.8}$$

The functions π_{11}, π_{22}, π_{33} are called *abnormalities* while the functions π_{ab}, $a \neq b$ are given the generic title of *curvatures*. A corresponding set of pi functions is given by

$$\Pi_{AB} = \underset{\sim}{E}_B \cdot \text{curl } \underset{\sim}{E}_A \;. \tag{1.9}$$

*Throughout the work the appendage, *etc.* will indicate that three equations obtained by cyclic permutation of the subscripts, is implied.

The functions π_{ab} and $\gamma^c_{bc} = - \gamma^c_{ba}$ are related as follows:

$$\begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} = \begin{bmatrix} \gamma^1_{32} - \gamma^1_{23} & \gamma^1_{13} & \gamma^2_{11} \\ \gamma^3_{22} & \gamma^2_{13} - \gamma^2_{31} & \gamma^2_{21} \\ \gamma^3_{32} & \gamma^1_{33} & \gamma^3_{21} - \gamma^3_{12} \end{bmatrix} \tag{1.10}$$

$$2\gamma^1_{32} = \pi_{11} + \pi_{22} - \pi_{33}\,, \qquad 2\gamma^3_{21} = \pi_{33} + \pi_{11} - \pi_{22}\,,$$

$$2\gamma^2_{13} = \pi_{22} + \pi_{33} - \pi_{11}\,.$$

Similar relations hold for the upper case variables.

The condition curl grad F = $\underset{\sim}{0}$ applied to the scalar field F, yields, by (1.8) the basic commutation formulae*

$$\frac{\delta^2 F}{\delta e_3 \delta e_2} - \frac{\delta^2 F}{\delta e_2 \delta e_3} = \pi_{11} \frac{\delta F}{\delta e_1} + \pi_{21} \frac{\delta F}{\delta e_2} + \pi_{31} \frac{\delta F}{\delta e_3}\,, \textit{ etc.} \tag{1.11}$$

*We use the symbol $\frac{\delta F}{\delta e_a}$ to denote the component $\underset{\sim}{e}_a \cdot$ grad F. Then $\frac{\delta^2 F}{\delta e_a \delta e_b} = \underset{\sim}{e}_a \cdot$ grad ($e_b \cdot$ grad F), and so on.

We indicate two symmetry conditions which must be satisfied by all the general relations derived in the analysis. These conditions are thus extremely important as checks for the equations.*

1. *The reflection condition*

If the unit vector $\underset{\sim}{e}_1$ is changed to $-\underset{\sim}{e}_1$, while $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ are unaltered, the basis comprising the unit proper vectors of $\underset{\sim}{c}$ and $\underset{\sim}{c}^{-1}$ changes from a right-handed to a left-handed system. However, $\underset{\sim}{c}$ and $\underset{\sim}{c}^{-1}$ are unaltered. The operator curl changes to minus curl. From (1.18) we obtain the following transformation

$$\underset{\sim}{e}_1^* \to -\underset{\sim}{e}_1\,, \quad \underset{\sim}{e}_2^* \to \underset{\sim}{e}_2\,, \quad \underset{\sim}{e}_3^* \to \underset{\sim}{e}_3\,, \quad \pi_{11}^* \to -\pi_{11}\,,$$

$$\pi_{22}^* \to -\pi_{22}\,, \quad \pi_{33}^* \to -\pi_{33}\,, \quad \pi_{23}^* \to -\pi_{23}\,, \quad \pi_{32}^* \to -\pi_{32}\,,$$

$$\pi_{12}^* \to \pi_{12}\,, \quad \pi_{21}^* \to \pi_{21}\,, \quad \pi_{13}^* \to \pi_{13}\,, \quad \pi_{31}^* \to \pi_{31}\,, \ etc. \tag{1.12}$$

The general conditions derived must allow this transformation.

*The conditions do not apply to particular solutions. For example, in the Appendix we derive a condition

$$\left(\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right)\left(\pi_{11} + \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}\right)\left(\pi_{22} + \frac{\alpha_{31}}{\alpha_{32}}\pi_{11}\right) = 0\,. \tag{A.22}$$

We may say that *one* of the factors is zero, but *not* that all three are zero, even though the other two follow from the first by applying the rotation condition.

2. *The rotation condition*

The deformation tensors $\underset{\sim}{c}$ and $\underset{\sim}{c}^{-1}$ are unaltered if one rotates the axes ninety degrees about $\underset{\sim}{e}_3$, for example, and interchanges σ_1^2 and σ_2^2. The general conditions must be invariant under the transformation

$$e_2^* \rightarrow -\underset{\sim}{e}_1 , \quad \underset{\sim}{e}_1^* \rightarrow \underset{\sim}{e}_2 , \quad \underset{\sim}{e}_3^* \rightarrow \underset{\sim}{e}_3 , \quad \pi_{11}^* \rightarrow \pi_{22} ,$$

$$\pi_{22}^* \rightarrow \pi_{11} , \quad \pi_{33}^* \rightarrow \pi_{33} , \quad \pi_{12}^* \rightarrow -\pi_{21} , \quad \pi_{21}^* \rightarrow -\pi_{12} ,$$

$$\pi_{23}^* \rightarrow -\pi_{13} , \quad \pi_{32}^* \rightarrow -\pi_{31} , \quad \pi_{31}^* \rightarrow \pi_{32} , \quad \pi_{13}^* \rightarrow \pi_{23} ,$$

$$\sigma_1^{*2} \rightarrow \sigma_2^2 , \quad \sigma_2^{*2} \rightarrow \sigma_1^2 , \quad \sigma_3^{*2} \rightarrow \sigma_3^2 , \quad etc. \tag{1.13}$$

These relations are easily derivable from (1.8).

The deformation tensors $\underset{\sim}{C}$ and $\underset{\sim}{c}$ are metric tensors in Euclidean spaces. The Riemann curvature tensor calculated with $\underset{\sim}{C}$ or $\underset{\sim}{c}$ as metric must vanish. We obtain these conditions by a direct means first suggested by Yin.

It follows from (1.1) that the bases $\underset{\sim}{e}_a$ and $\underset{\sim}{E}_A$ are related through the deformation by*

$$\underset{\sim}{e}_{\underline{a}} \cdot d\underset{\sim}{x} = \underset{\sim}{e}_{\underline{a}} \cdot \underset{\sim}{F}\, d\underset{\sim}{X} = \sigma_{\underline{a}}\, \underset{\sim}{E}_{\underline{a}} \cdot d\underset{\sim}{X} . \tag{1.14}$$

When (1.5) holds, for (1.14) to be integrable one must have

*The underscore indicates that no sum is implied.

$$\Pi_{ab} = \frac{\pi_{\underline{ab}}}{\sigma_{\underline{a}}\sigma_{\underline{b}}} , \tag{1.15}$$

and

$$\frac{\delta \Pi_{ab}}{\delta E_{\underline{c}}} = \sigma_{\underline{c}} \frac{\delta \Pi_{ab}}{\delta e_{\underline{c}}} = \frac{\sigma_{\underline{c}}}{\sigma_{\underline{a}}\sigma_{\underline{b}}} \frac{\delta \pi_{\underline{ab}}}{\delta e_{\underline{c}}} . \tag{1.16}$$

Since the bases $\underset{\sim}{e}_a$ and $\underset{\sim}{E}_A$ are each embedded in Euclidean spaces the curvature tensors based on the connections γ^b_{ac} and Γ^B_{AC} **must vanish.** Thus

$$r_{cbd}^{\cdot\cdot\cdot a} = \frac{\delta}{\delta e_c}\gamma^a_{bd} - \frac{\delta}{\delta e_b}\gamma^a_{cd} + \gamma^a_{ce}\gamma^e_{bd} - \gamma^a_{be}\gamma^e_{cd} + \left(\gamma^e_{bc} - \gamma^e_{cb}\right)\gamma^a_{ed} = 0 \tag{1.17}$$

and

$$R_{CBD}^{\cdot\cdot\cdot A} = \frac{\delta}{\delta E_C}\Gamma^A_{BD} - \frac{\delta}{\delta E_B}\Gamma^A_{CD} + \Gamma^A_{CE}\Gamma^E_{BD} - \Gamma^A_{BE}\Gamma^E_{CD} + \left(\Gamma^E_{BC} - \Gamma^E_{CB}\right)\Gamma^A_{ED} = 0 . \tag{1.18}$$

The conditions (1.17) and (1.18) may be expressed in **terms of** π_{ab} and Π_{AB} by (1.10). The condition involving Π_{AB} and $\frac{\delta\Pi_{AB}}{\delta E_C}$ **is then** transformed through (1.15) and (1.16) into a corresponding **condition** involving π_{ab}, $\frac{\delta\pi_{ab}}{\delta e_c}$ and σ_1^2 , σ_2^2 , σ_3^2 . The method is **presented** fully in (1975 [1], p. 117, 118).*

There are nine independent conditions represented by each of (1.17) and (1.18). These occur in three sets of cyclic conditions a specimen of each set being given by

$$r_{232}^{\cdots 3} = 0 , \qquad r_{232}^{\cdots 1} = 0 , \qquad r_{312}^{\cdots 1} = 0 \tag{1.19}$$

$$R_{232}^{\cdots 3} = 0 , \qquad R_{232}^{\cdots 1} = 0 , \qquad R_{312}^{\cdots 1} = 0 \tag{1.20}$$

We write

$$\alpha_{rs} \overset{\text{def}}{=} 1 - \frac{\sigma_r^2}{\sigma_s^2} \neq 0 , \tag{1.21}$$

and note the identities

*In (1975 [1]) the curvatures were immediately expressed as gradient functions in accordance with the equilibrium conditions. This obscures the purely geometrical origin of many of the relations. Also in (1975 [1]) the condition (1.33) was not derived as a geometrical condition from (1.17) and (1.18). It happens, as will be seen, that this same condition also follows from the equilibrium conditions. In (1975 [1]) it was obtained as a consequence of the equilibrium conditions. Here we indicate the geometrical origin of the conditions first to exhibit the geometrical problem as a separate problem and then to indicate the essential weakness of the equilibrium conditions.

$$\frac{\alpha_{12}\ \alpha_{23}\ \alpha_{31}}{\alpha_{21}\ \alpha_{32}\ \alpha_{13}} \equiv -1 \tag{1.22}$$

$$\frac{\alpha_{13}}{\alpha_{12}} \equiv 1 - \frac{\alpha_{23}}{\alpha_{21}}\ , \quad etc. \tag{1.23}$$

From 1975 [1] equations (2.5) and (2.6), we have, corresponding to $r_{232}^{\cdot\cdot\cdot 3} = 0$ and $R_{232}^{\cdot\cdot\cdot 3} = 0$,

$$\frac{\delta}{\delta e_3}\pi_{21} = \pi_{21}^2 - \frac{\alpha_{12}}{\alpha_{32}}\pi_{32}\pi_{23} + \xi_{321}\ , \quad etc.\ , \tag{1.24}$$

where

$$\xi_{321} = -\frac{\pi_{11}\pi_{22}}{2} + \frac{\alpha_{12}}{\alpha_{32}}\frac{\pi_{22}\pi_{33}}{2} + \frac{3}{4}\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}^2 - \frac{1}{4}\left[\frac{\alpha_{32}}{\alpha_{31}} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{31}\alpha_{23}}\right]\pi_{22}^2$$

$$- \frac{1}{4}\frac{\alpha_{13}}{\alpha_{32}}\pi_{33}^2\ , \quad etc. \tag{1.25}$$

and

$$\frac{\delta}{\delta e_2}\pi_{31} = -\pi_{31}^2 + \frac{\alpha_{13}}{\alpha_{23}}\pi_{32}\pi_{23} + \eta_{231}\ , \quad etc.\ , \tag{1.26}$$

where*

$$\eta_{231} = \frac{\pi_{11}\pi_{33}}{2} - \frac{\alpha_{13}}{\alpha_{23}}\frac{\pi_{22}\pi_{33}}{2} - \frac{3}{4}\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2 + \frac{1}{4}\frac{\alpha_{13}}{\alpha_{12}}\left[\frac{\alpha_{13}}{\alpha_{23}} - \frac{\alpha_{32}}{\alpha_{31}}\right]\pi_{33}^2$$

$$+ \frac{1}{4}\frac{\alpha_{12}}{\alpha_{23}}\pi_{22}^2 \text{ , } etc. \tag{1.27}$$

The conditions (1.26), (1.27) are dual to (1.24), (1.25) in that one set may be obtained from the other by applying the rotation condition (1.13).

From 1975 [1], equation (2.13), we have, corresponding to $r_{232}^{\cdot\cdot\cdot 1} = 0$ and $R_{232}^{\cdot\cdot\cdot 1} = 0$,

$$\frac{\delta}{\delta e_2}\pi_{11} - 2\pi_{13}\pi_{11} - \frac{\alpha_{23}}{\alpha_{21}}\left(\frac{\delta}{\delta e_2}\pi_{33} + 2\pi_{31}\pi_{33}\right) = 0, \; etc. \tag{1.28}$$

The set of three equations (1.28) transforms into itself under the

*To reconcile the forms (1.25) and (1.27) to 1975 [1] equations (2.5) and (2.6), one may readily verify that

$$\frac{1}{\alpha_{32}}\left(1 - \frac{\sigma_1^2\sigma_3^2}{\sigma_2^4}\right) = \frac{\alpha_{32}}{\alpha_{31}} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{31}\alpha_{23}} \text{ ,}$$

and

$$\frac{1}{\alpha_{23}}\left(1 - \frac{\sigma_1^2\sigma_2^2}{\sigma_3^4}\right) = \frac{\alpha_{13}}{\alpha_{12}}\left(\frac{\alpha_{13}}{\alpha_{23}} - \frac{\alpha_{32}}{\alpha_{31}}\right) .$$

rotation condition (1.13).

We now introduce

$$u_1 \overset{\text{def}}{=} \frac{\delta}{\delta e_2}\pi_{33} + 2\pi_{31}\pi_{33}\ , \quad etc., \tag{1.29}$$

so that by (1.28)

$$\frac{\delta}{\delta e_2}\pi_{11} - 2\pi_{13}\pi_{11} = \frac{\alpha_{23}}{\alpha_{21}}\ u_1\ , \ etc. \tag{1.30}$$

We note that under the rotation condition transformation (1.13), the variables u_a transform as follows

$$u_1^* \to -\frac{\alpha_{12}}{\alpha_{13}}\,u_3\ , \qquad u_2^* \to \frac{\alpha_{31}}{\alpha_{32}}\,u_2\ , \qquad u_3^* \to \frac{\alpha_{23}}{\alpha_{21}}\,u_1\ . \tag{1.31}$$

The second condition obtained from $r_{232}^{\cdot\cdot\cdot 1} = 0$ and $R_{232}^{\cdot\cdot\cdot 1} = 0$, obtained from 1975 (1) equation (2.14) and (1.30), is

$$\frac{\delta}{\delta e_3}\pi_{23} = -\pi_{22}\pi_{13} + \pi_{21}\,(\pi_{23} + \pi_{32}) + \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}\,u_1\ , \ etc. \tag{1.32}$$

Transforming (1.32) by (1.13) and (1.31), and using (1.22) we obtain the dual set

$$\frac{\delta}{\delta e_3}\pi_{13} = \pi_{11}\pi_{23} - \pi_{12}\,(\pi_{13} + \pi_{31}) - \frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\,u_3,\ etc. \tag{1.33}$$

We have the three conditions div curl $\underset{\sim}{e}_a = 0$, so that, by (1.8),

$$\frac{\delta \pi_{11}}{\delta e_1} + \pi_{11} \operatorname{div} \underset{\sim}{e}_1 + \frac{\delta}{\delta e_2} \pi_{12} + \pi_{12} \operatorname{div} \underset{\sim}{e}_2$$

$$+ \frac{\delta}{\delta e_3} \pi_{13} + \pi_{13} \operatorname{div} \underset{\sim}{e}_3 = 0 \; , \quad etc. \tag{1.34}$$

and from (1.6) and (1.10)

$$\operatorname{div} \underset{\sim}{e}_1 = \pi_{23} - \pi_{32} \; , \quad etc. \tag{1.35}$$

We verify that (1.33) is indeed a geometrical condition corresponding to $r_{312}^{\cdot\cdot\cdot 1} = 0$. From (1.7), (1.10) and (1.17) we have for $r_{312}^{\cdot\cdot\cdot 1} = 0$

$$- \frac{\delta}{\delta e_3} \pi_{13} - \frac{1}{2} \frac{\delta}{\delta e_1} (\pi_{11} + \pi_{22} - \pi_{33}) - \pi_{32}(\pi_{33} - \pi_{11})$$

$$- \pi_{12}(\pi_{31} + \pi_{13}) - \pi_{22}\pi_{23} = 0 \; . \tag{1.36}$$

Substituting for $\frac{\delta}{\delta e_1} \pi_{11}$, $\frac{\delta}{\delta e_1} \pi_{22}$, $\frac{\delta}{\delta e_1} \pi_{33}$ from (1.34), (1.29) and (1.30) respectively, using (1.32) for $\frac{\delta}{\delta e_2} \pi_{12}$ and using the expressions (1.35) we regain the condition (1.33).*

*The nine independent conditions implied by (1.17) include the three conditions (1.34). The conditions (1.5), (1.15) and (1.16) mean that condition DIV CURL $\underset{\sim}{E}_\Lambda = 0$, computed for the images of the proper numbers before deformation, implies div curl $\underset{\sim}{e}_a = 0$. We thus get fifteen rather than eighteen independent compatibility conditions for the deformation. This was noted by Yin. These conditions are the sets (1.24), (1.26), (1.28), (1.32) and (1.33), and (1.37)

with [illegible] eliminated

2. GEOMETRY OF ISOCHORIC DEFORMATIONS WITH CONSTANT PROPER NUMBERS. THREE SMALL THEOREMS.

Eliminating u_3 from the sets (1.32) and (1.33) we obtain

$$\frac{\delta}{\delta c_2}\pi_{12} + \frac{\delta}{\delta c_3}\pi_{13} + \pi_{12}\pi_{31} - \pi_{13}\pi_{21} - \pi_{11}(\pi_{23} - \pi_{32}) = 0, \; etc. \tag{2.1}$$

From (1.34) and (2.1) using (1.35) we get

$$\frac{\delta\pi_{11}}{\delta c_1} + 2\pi_{11}\,\mathrm{div}\,\underset{\sim}{c}_1 = 0 \;, \; etc., \tag{2.2}$$

or

$$\mathrm{div}\,(\pi_{11}^{1/2}\,\underset{\sim}{c}_1) = 0 \;, \; etc. \qquad .$$

One has

Theorem 2.1. The vector fields of the unit proper vectors $\underset{\sim}{e}_a$ are complex-solenoidal fields:

$$\underset{\sim}{c}_a = \frac{1}{\pi_{11}^{1/2}}\,\mathrm{curl}\,\underset{\sim}{p}_a \;.$$

By (1.15) and (1.16) an analogous theorem holds for the unit proper vectors $\underset{\sim}{E}_A$.

We now note that

$$\mathrm{div}(\underset{\sim}{c}_1\underset{\sim}{c}_1) = \mathrm{div}\,\underset{\sim}{c}_1\,\underset{\sim}{c}_1 + \pi_{13}\underset{\sim}{c}_2 - \pi_{12}\underset{\sim}{c}_3 \;, \; etc. \tag{2.3}$$

and that the three conditions (2.1) express

$$\underset{\sim}{e}_1 \cdot \operatorname{curl} \operatorname{div}(\underset{\sim}{e}_1 \underset{\sim}{e}_1) = 0 \;, \quad \textit{etc.} \tag{2.4}$$

We have

Theorem 2.2. The condition (2.4) is a geometrical result for isochoric deformations with constant distinct proper numbers.

The significance of Theorem 2.2 is that it tells us that the equilibrium conditions to be considered in the next section give only three cyclic scalar conditions, rather than six as they appear to do at first sight. It emphasizes the ultimate weakness of the equilibrium conditions.

From (1.29), (1.30) and (2.2)[1] we have for non-vanishing abnormalities

$$\begin{aligned} &\frac{1}{2}\left(\frac{u_3}{\pi_{22}} + \frac{\alpha_{12}}{\alpha_{13}}\,\frac{u_3}{\pi_{33}}\right) - 2 \operatorname{div} \underset{\sim}{e}_1 \\ &\quad = \frac{\delta}{\delta e_1} \log \pi_{22}^{1/2} + \pi_{23} + \frac{\delta}{\delta e_1} \log \pi_{33}^{1/2} - \pi_{32} \\ &\qquad + \frac{\delta}{\delta e_1} \log \pi_{11}^{1/2} - \operatorname{div} \underset{\sim}{e}_1 \;. \end{aligned} \tag{2.5}$$

From (1.35) and (2.5) there follows:

Theorem 2.3. For isochoric deformations with constant distinct proper numbers, and for which the abnormalities of the vector fields of $\underset{\sim}{e}_a$ do not vanish, one has

$$\left[-4 \operatorname{div} \underset{\sim}{e}_1 + \frac{\lambda_1 u_3}{\pi_{22}\pi_{33}}\right] \underset{\sim}{e}_1 + \left[-4 \operatorname{div} \underset{\sim}{e}_2 + \frac{\lambda_2 u_1}{\pi_{33}\pi_{11}}\right] \underset{\sim}{e}_2$$

$$+ \left[-4 \operatorname{div} \underset{\sim}{e}_3 + \frac{\lambda_3 u_2}{\pi_{11}\pi_{22}}\right] \underset{\sim}{e}_3$$

$$= \operatorname{grad} \log \pi_{11}\pi_{22}\pi_{33} \,, \tag{2.6}$$

where

$$\lambda_1 \overset{\text{def}}{=} \left(\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}} \pi_{22}\right) \,, \quad \textit{etc.} \tag{2.7}$$

3. INTRODUCTION OF THE EQUILIBRIUM CONDITIONS

When the proper numbers σ_1^2, σ_2^2, σ_3^2 of $\underset{\sim}{c}^{-1}$ are constant, the equilibrium conditions for the incompressible perfectly elastic material reduce to

$$\text{curl div } \underset{\sim}{c} = \underset{\sim}{0} \ ,$$

and

$$\text{curl div } \underset{\sim}{c}^{-1} = \underset{\sim}{0} \ . \tag{3.1}$$

For distinct proper numbers these conditions require that

$$\text{curl div } (\underset{\sim}{c}_1\underset{\sim}{c}_1) = \underset{\sim}{0} \ , \quad \text{curl div } (\underset{\sim}{c}_2\underset{\sim}{c}_2) = \underset{\sim}{0} \ ,$$
$$\text{curl div } (\underset{\sim}{c}_3\underset{\sim}{c}_3) = \underset{\sim}{0} \ , \tag{3.2}$$

where

$$\underset{\sim}{c}_1\underset{\sim}{c}_1 + \underset{\sim}{c}_2\underset{\sim}{c}_2 + \underset{\sim}{c}_3\underset{\sim}{c}_3 = \underset{\sim}{1} \ . \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$(\text{div } \underset{\sim}{c}_1)\underset{\sim}{c}_1 + \pi_{13}\underset{\sim}{c}_2 - \pi_{12}\underset{\sim}{c}_3 = \text{grad } \theta_1 \ , \quad \textit{etc.} \tag{3.4}$$

where

$$\theta_1 + \theta_2 + \theta_3 = \text{constant} \ .$$

The condition $(3.4)^2$ of course follows from (1.35).

Taking the curl of (3.4)[1] we obtain two independent conditions

$$\frac{\delta}{\delta e_1}\pi_{13} = \frac{\delta}{\delta e_2}\operatorname{div}\underset{\sim}{e}_1 - \pi_{13}(2\pi_{23} - \pi_{32}) + \pi_{12}\pi_{33}\ , \ etc.,$$

$$\frac{\delta}{\delta e_1}\pi_{12} = -\frac{\delta}{\delta e_3}\operatorname{div}\underset{\sim}{e}_1 - \pi_{12}(\pi_{23} - 2\pi_{32}) - \pi_{13}\pi_{22}, \ etc. \qquad (3.5)$$

Equations (3.5) are dual, in that they can be derived from each other through the rotation condition (1.13).

It is important to note that the self-dual set of conditions (2.1), which are consequences of the geometrical conditions only can be obtained from the conditons (3.5). Thus, the equilibrium conditions really only give one set of scalar equations that are independent of the geometrical conditions. This is evident also from equations (2.4) and (3.2).

We now introduce the subsidiary variables

$$x_1 \overset{\text{def}}{=} \frac{\delta}{\delta e_2}\operatorname{div}\underset{\sim}{e}_3 + \pi_{31}\operatorname{div}\underset{\sim}{e}_3\ , \qquad etc.,$$

$$y_1 \overset{\text{def}}{=} \frac{\delta}{\delta e_3}\operatorname{div}\underset{\sim}{e}_2 - \pi_{21}\operatorname{div}\underset{\sim}{e}_2\ , \qquad etc. \qquad (3.6)$$

Under the rotation condition transformation (1.13) these variables x_a and y_a transform as follows

$$\overset{*}{x}_1 \rightarrow -y_2\ , \quad \overset{*}{y}_1 \rightarrow -x_2\ , \quad \overset{*}{x}_2 \rightarrow y_1\ , \quad \overset{*}{y}_2 \rightarrow x_1\ ,$$

$$\overset{*}{x}_3 \rightarrow -y_3\ , \quad \overset{*}{y}_3 \rightarrow -x_3\ . \qquad (3.7)$$

From (3.5) we obtain the dual sets of conditions

$$\frac{\delta}{\delta e_1}\pi_{13} = y_3 - \pi_{13}\pi_{23} + \pi_{12}\pi_{33}\ , \quad etc.,$$

$$\frac{\delta}{\delta e_1}\pi_{12} = - x_2 + \pi_{12}\pi_{32} - \pi_{13}\pi_{22}\ , \ etc.$$

We remember that (3.8) and (3.9) do imply the use of the condition.

From (1.35) and (3.8)[1]

$$\frac{\delta}{\delta e_2}\pi_{21} = \frac{\delta}{\delta e_2}\pi_{12} - \frac{\delta}{\delta e_2}\operatorname{div} \underset{\sim}{e}_3 = y_1 - \pi_{21}\pi_{31} + \pi_2$$

whence, by (3.6) and (1.35)

$$\frac{\delta}{\delta e_2}\pi_{12} = x_1 + y_1 - \pi_{12}\pi_{31} + \pi_{23}\pi_{11}$$

so that by (1.32), we obtain the self dual set of condit

$$x_1 + y_1 + \pi_{11}(\pi_{23} + \pi_{32}) - \pi_{12}(\pi_{31} + \pi_{13})$$
$$- \pi_{13}\pi_{21} - \frac{\alpha_{32}}{2\alpha_{31}}u_3 = 0 \quad .$$

From (1.29), (1.30) and (3.4) we obtain

$$\frac{u_1}{2\pi_{33}} = \frac{\delta}{\delta e_2}\ (\log \pi_{33}^{1/2} - \theta_3)\ , \quad etc.,$$

α_1

$$\frac{1}{2}\,\frac{\alpha_{12}}{\alpha_{13}}\,\frac{u_3}{\pi_{33}} = \frac{\delta}{\delta e_1}\,\left(\log\,\pi_{33}^{1/2} - \theta_3\right)\ , \quad etc., \tag{3.13}$$

while by $(2.2)^1$ and (3.4)

$$\frac{\delta}{\delta e_1}\,\left(\log\,\pi_{11}^{1/2} - \theta_1\right) = -2\,\mathrm{div}\,e_1\ , \quad etc. \tag{3.14}$$

From (1.11) one has

$$\left[\frac{\delta^2}{\delta e_2 \delta e_1} - \frac{\delta^2}{\delta e_1 \delta e_2}\right]\ \left[\log\,\pi_{11}^{1/2} - \theta_1\right]$$

$$= \left[\pi_{13}\,\frac{\delta}{\delta e_1} + \pi_{23}\,\frac{\delta}{\delta e_2} + \pi_{33}\,\frac{\delta}{\delta e_3}\right]\ \left[\log\,\pi_{11}^{1/2} - \theta_1\right]$$

so that from (3.12), (3.13) and (3.14),

$$\frac{\delta}{\delta e_2}\,(2\,\mathrm{div}\,\underset{\sim}{e}_1) - \frac{\delta}{\delta e_1}\left(\frac{\alpha_{23}}{\alpha_{21}}\,\frac{u_1}{2\pi_{11}}\right)$$

$$= -\,2\pi_{13}\,\mathrm{div}\,\underset{\sim}{e}_1 + \pi_{23}\,\frac{\alpha_{23}}{\alpha_{21}}\,\frac{u_1}{2\pi_{11}} + \pi_{33}\,\frac{u_2}{2\pi_{11}}\ ,$$

which, using (1.35) and $(3.6)^2$, reduces to

$$\frac{\alpha_{23}}{\alpha_{21}}\,\frac{\delta u_1}{\delta e_1} = -\,4\pi_{11}\gamma_3 - (3\pi_{23} - 2\pi_{32})\,\frac{\alpha_{23}}{\alpha_{21}}\,u_1 - \pi_{33}\,u_2\ , \quad etc. \tag{3.15}$$

Again from

$$\left[\frac{\delta^2}{\delta e_3 \delta e_2} - \frac{\delta^2}{\delta e_2 \delta e_3}\right] \left[\log \pi_{33}^{1/2} - \theta_3\right]$$

$$= \left[\pi_{12} \frac{\delta}{\delta e_1} + \pi_{21} \frac{\delta}{\delta e_2} + \pi_{31} \frac{\delta}{\delta e_3}\right] \left[\log \pi_{33}^{1/2} - \theta_3\right]$$

and (1.35), (3.12), (3.13), (3.14) and (3.6)[1], one obtains,

$$\frac{\delta u_1}{\delta e_3} = -4\, \pi_{33} x_1 + (3\pi_{21} - 2\pi_{12}) u_1 + \pi_{11} \frac{\alpha_{12}}{\alpha_{13}} u_3 \ , \ etc. \tag{3.16}$$

The conditions (3.16) may otherwise be obtained from (3.14) by applying the rotation transformations (1.13), (1.31) and (3.7).

Finally, treating the commutation

$$\left[\frac{\delta^2}{\delta e_3 \delta e_2} - \frac{\delta^2}{\delta e_2 \delta e_3}\right] \left[\log \pi_{11}^{1/2} - \theta_1\right]$$

$$= \left[\pi_{11} \frac{\delta}{\delta e_1} + \pi_{21} \frac{\delta}{\delta e_2} + \pi_{31} \frac{\delta}{\delta e_3}\right] \left[\log \pi_{11}^{1/2} - \theta_1\right]$$

in a similar manner one obtains the set

$$\frac{\alpha_{23}}{\alpha_{21}} \frac{\delta u_1}{\delta e_3} - \frac{\delta u_2}{\delta e_2} = -4\pi_{11}^2 \operatorname{div} \underset{\sim}{e}_1 + (\pi_{21} - 2\pi_{12}) \frac{\alpha_{23}}{\alpha_{21}} u_1$$

$$+ (\pi_{31} - 2\pi_{13})\, u_2 \ , \quad etc. \tag{3.17}$$

Eliminating the gradients $\frac{\delta u_1}{\delta c_3}$ and $\frac{\delta u_2}{\delta c_2}$ from (3.17) by means of (3.15) and (3.16), and using (1.22), we get the integrals

$$\pi_{21} \frac{\alpha_{23}}{\alpha_{21}} u_1 + \pi_{31} u_2 - 2 \frac{\alpha_{23}}{\alpha_{21}} \pi_{33} x_1 + 2 \frac{\alpha_{32}}{\alpha_{31}} \pi_{22} y_1$$

$$+ 2\pi_{11}^2 \operatorname{div} e_1 = 0 , \quad \textit{etc.} \tag{3.18}$$

We may now solve (3.11) and (3.18) to give the x_α and y_α in terms of u_α. One obtains

$$\mu_1 x_1 = \frac{1}{2} \left(\frac{\alpha_{23}}{\alpha_{21}} \pi_{21} u_1 + \pi_{31} u_2 + \frac{\alpha_{32}\alpha_{32}}{\alpha_{31}\alpha_{31}} \pi_{22} u_3 \right)$$

$$- \frac{\alpha_{32}}{\alpha_{31}} \pi_{22} f_1 + \pi_{11}^2 \operatorname{div} \underset{\sim}{e}_1 ,$$

$$\mu_1 y_1 = - \frac{1}{2} \left(\frac{\alpha_{23}}{\alpha_{21}} u_1 + \pi_{31} u_2 - \frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{31}} \pi_{33} u_3 \right)$$

$$- \frac{\alpha_{23}}{\alpha_{21}} \pi_{33} f_1 - \pi_{11}^2 \operatorname{div} \underset{\sim}{e}_1 , \quad \textit{etc.} \tag{3.19}$$

*The conditions (3.18) are the statement that the curl of the left hand side (2.6) vanishes. In the subsequent analysis we shall take directional derivatives of (3.18). We must avoid the identity represented by div curl = 0.

where

$$\mu_1 \overset{\text{def}}{=} \frac{\alpha_{32}}{\alpha_{31}}\pi_{22} + \frac{\alpha_{23}}{\alpha_{21}}\pi_{33} = \frac{\alpha_{23}}{\alpha_{21}}\left(\pi_{33} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right) ,$$

$$\text{by } (1.22) , \quad etc., \tag{3.20}$$

and

$$f_1 = \pi_{11}(\pi_{23} + \pi_{32}) - \pi_{12}(\pi_{31} + \pi_{13}) - \pi_{13}\pi_{21} , \quad etc. \tag{3.21}$$

4. THE GRADIENTS OF x_a, y_a

The twenty-seven relations given by the sets (1.24), (1.26), (1.29), (1.30), (1.32), (1.33), (2.2), (3.8) and (3.9) give the three components of the gradients of the nine functions π_{ab} in terms of the variables π_{ab}, x_a, y_a and u_a. The six conditions (3.15) and (3.16) give the six gradients $\frac{\delta u_1}{\delta e_1}$, $\frac{\delta u_2}{\delta e_2}$, $\frac{\delta u_3}{\delta e_3}$, $\frac{\delta u_1}{\delta e_3}$, $\frac{\delta u_2}{\delta e_1}$, $\frac{\delta u_3}{\delta e_2}$ in terms of these variables.

In this chapter we develop eighteen relations giving the three components of the gradients of the six functions x_a, y_a in terms of the variables π_{ab}, x_a, y_a, u_a and the three gradients, $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$. It is apparent that no immediate expressions for these three gradients are available. We shall thus include them as additional variables, to be eliminated subsequently.

From (3.9) one has

$$\frac{\delta x_3}{\delta e_1} = -\frac{\delta^2}{\delta e_1 \delta e_2}\pi_{23} + \frac{\delta}{\delta e_1}\left(\pi_{13}\pi_{23} - \pi_{21}\pi_{33}\right)$$

$$= -\frac{\delta^2}{\delta e_2 \delta e_1}\pi_{23} + 2\pi_{13}\frac{\delta}{\delta e_1}\pi_{23} + \pi_{23}\left(\frac{\delta}{\delta e_2}\pi_{23} + \frac{\delta}{\delta e_1}\pi_{13}\right)$$

$$+ \pi_{33}\left(\frac{\delta}{\delta e_3}\pi_{23} - \frac{\delta}{\delta e_1}\pi_{21}\right) - \pi_{21}\frac{\delta}{\delta e_1}\pi_{33}\text{ , by (1.11).}$$

Making the appropriate substitutions for the gradient terms we obtain*

*Note that terms $\frac{\delta}{\delta e_1}\pi_{23}$ and $\frac{\delta^2}{\delta e_2 \delta e_1}\pi_{23}$ may be expressed in terms of u_a and π_{ab} through (1.24) to (1.27). For reasons of economy of space such substitutions will be delayed until later in the analysis.

$$\frac{\delta x_3}{\delta e_1} = -\frac{\delta^2}{\delta e_2 \delta e_1}\pi_{23} + 2\pi_{13}\frac{\delta}{\delta e_1}\pi_{23} + \pi_{23}(y_3 - x_3) + \frac{\alpha_{13}}{\alpha_{12}}\pi_{33}u_1 - \frac{\alpha_{12}}{\alpha_{13}}\pi_{21}u_3$$

$$+ \pi_{33}\left[-\pi_{22}(\pi_{13} + \pi_{31}) + \pi_{23}(2\pi_{12} + \pi_{21}) - \pi_{21}\pi_{32}\right] \text{, } etc. \quad (4.1)$$

From $(3.6)^2$ and (1.35), one has

$$\frac{\delta y_3}{\delta e_1} = \frac{\delta^2}{\delta e_1 \delta e_2}(\pi_{23} - \pi_{32}) - \frac{\delta}{\delta e_1}(\pi_{13}\pi_{23} - \pi_{13}\pi_{32}) .$$

Commuting the first term by (1.11), substituting for the gradients and reducing we obtain

$$\frac{\delta y_3}{\delta e_1} = \frac{\delta^2}{\delta e_2 \delta e_1}(\pi_{23} - \pi_{32}) - 2\pi_{13}\frac{\delta}{\delta e_1}(\pi_{23} - \pi_{32}) + (\pi_{32} - 2\pi_{23})y_3$$

$$- \pi_{33}x_2 \text{, } \quad etc. \quad (4.2)$$

A check on (4.1) and (4.2) is obtained as follows. If they are added we obtain an expression for $\frac{\delta}{\delta e_1}(x_3 + y_3)$, which may be verified by taking the appropriate gradient of (3.10).

From $(3.6)^1$ and (1.35) one has

$$\frac{\delta x_3}{\delta e_2} = \frac{\delta^2}{\delta e_2 \delta e_1}(\pi_{31} - \pi_{13}) + \frac{\delta}{\delta e_2}(\pi_{23}\pi_{31} - \pi_{23}\pi_{13}) .$$

Commuting the first term by (1.11), substituting for the gradients and reducing, we obtain

$$\frac{\delta x_3}{\delta e_2} = \frac{\delta^2}{\delta e_1 \delta e_2} (\pi_{31} - \pi_{13}) + 2\pi_{23} \frac{\delta}{\delta e_2} (\pi_{31} - \pi_{13}) - (\pi_{31} - 2\pi_{13}) x_3$$

$$+ \pi_{33} y_1 \ , \quad etc. \tag{4.3}$$

From (3.8) one has

$$\frac{\delta y_3}{\delta e_2} = \frac{\delta^2}{\delta e_2 \delta e_1} \pi_{13} + \frac{\delta}{\delta e_2} (\pi_{13}\pi_{23} - \pi_{12}\pi_{33}) \ ,$$

which similarly leads to

$$\frac{\delta y_3}{\delta e_2} = \frac{\delta^2}{\delta e_1 \delta e_2} \pi_{13} + 2\pi_{23} \frac{\delta}{\delta e_2} \pi_{13} + \pi_{13}(y_3 - x_3) - \pi_{12} u_1$$

$$- \frac{\alpha_{32}}{\alpha_{31}} \pi_{33} u_3 + \pi_{33} \Big[\pi_{11} (\pi_{23} + \pi_{32}) - \pi_{13} (2\pi_{21} + \pi_{12})$$

$$+ \pi_{12}\pi_{31} \Big] \ , \qquad etc. \tag{4.4}$$

A check on (4.3) and (4.4) is obtained by taking the appropriate gradient of (3.10).

Again from (3.6) we have

$$\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2} = \left[\frac{\delta^2}{\delta e_3 \delta e_2} - \frac{\delta^2}{\delta e_2 \delta e_3} \right] \operatorname{div} \underset{\sim}{e}_1 - \frac{\delta}{\delta e_3} \left[\pi_{13} \operatorname{div} \underset{\sim}{e}_1 \right] - \frac{\delta}{\delta e_2} \left[\pi_{12} \operatorname{div} \underset{\sim}{e}_1 \right]$$

Applying the commutation formula (1.11), expanding, using (3.6) to eliminate the gradients of div $\underset{\sim}{e}_1$, and using (3.10), we obtain the set of relations

$$\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2} = \pi_{11} \left[\frac{\delta}{\delta e_1} \operatorname{div} \underset{\sim}{e}_1 - (\operatorname{div} \underset{\sim}{e}_1)^2\right] + \operatorname{div} \underset{\sim}{e}_2\, x_2$$

$$- \operatorname{div} \underset{\sim}{e}_3\, y_3 , \quad \textit{etc.} \tag{4.5}$$

We note that by eliminating $\frac{\delta}{\delta e_2} \pi_{12}$ between (2.1) and (3.10) we obtain the form

$$\frac{\delta}{\delta e_3} \pi_{13} = - (y_1 + x_1) + \pi_{21}\pi_{13} - \pi_{11}\pi_{32} , \quad \textit{etc.} \tag{4.6}$$

This result also follows directly from (1.33) and (3.11). From (3.8) and (4.6) we have

$$\frac{\delta y_3}{\delta e_3} + \frac{\delta}{\delta e_1} (y_1 + x_1) = \frac{\delta}{\delta e_3} \left[\frac{\delta}{\delta e_1} \pi_{13} + \pi_{13}\pi_{23} - \pi_{12}\pi_{33}\right]$$

$$+ \frac{\delta}{\delta e_1} \left[- \frac{\delta}{\delta e_3} \pi_{13} + \pi_{21}\pi_{13} - \pi_{11}\pi_{32}\right] .$$

Applying (1.11), expanding and eliminating $\frac{\delta}{\delta e_1} \pi_{13}$, $\frac{\delta}{\delta e_3} \pi_{13}$, $\frac{\delta}{\delta e_3} \pi_{23}$ and $\frac{\delta}{\delta e_1} \pi_{21}$ by (3.8), (4.6), (3.10) and (4.6) again, respectively, we obtain

$$\frac{\delta y_3}{\delta e_3} + \frac{\delta}{\delta e_1} (y_1 + x_1) = - \operatorname{div} \underset{\sim}{e}_1 (y_1 + x_1) - \operatorname{div} \underset{\sim}{e}_3\, y_3$$

$$- \pi_{11} \left[\frac{\delta}{\delta e_1} \pi_{32} - \pi_{32} \operatorname{div} \underset{\sim}{e}_1\right] - \pi_{22} \left[\frac{\delta}{\delta e_2} \pi_{13} - \pi_{13} \operatorname{div} \underset{\sim}{e}\right.$$

$$\left. - \pi_{33} \left[\frac{\delta}{\delta e_3} \pi_{12} - \pi_{12} \operatorname{div} \underset{\sim}{e}_3\right] , \quad \textit{etc.} \tag{4.7}$$

Taking the directional derivative of (3.11) with respect to $\underset{\sim}{e}_1$ and eliminating $\frac{\delta\pi_{11}}{\delta e_1}$, $\frac{\delta\pi_{12}}{\delta e_1}$, $\frac{\delta\pi_{13}}{\delta e_1}$, $\frac{\delta\pi_{31}}{\delta e_1}$ and $\frac{\delta\pi_{21}}{\delta e_1}$ by (2.2), (3.9), (3.8), (1.32) and (1.33) respectively, we obtain

$$\frac{\delta}{\delta e_1}(y_1 + x_1) = \frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1} - \pi_{11}\left[\frac{\delta}{\delta e_1}(\pi_{23} + \pi_{32}) - 2(\pi_{23}^2 - \pi_{32}^2)\right]$$

$$- (\pi_{31} + \pi_{13})x_2 + (\pi_{12} + \pi_{21})y_3 - \frac{\alpha_{13}}{2\alpha_{12}}\pi_{13}u_1$$

$$+ \frac{\alpha_{21}}{2\alpha_{23}}\pi_{12}u_2 - \pi_{22}\pi_{13}^2 + \pi_{33}\pi_{12}^2 + 2\pi_{12}\pi_{32}(\pi_{13} + \pi_{31})$$

$$- 2\pi_{13}\pi_{23}(\pi_{12} + \pi_{21}) \text{ , } etc. \tag{4.8}$$

Eliminating $\frac{\delta}{\delta e_1}(y_1 + x_1)$ between (4.7) and (4.8) and replacing $y_1 + x_1$ in (4.7) by the expression (3.11) we obtain, using (1.35)

$$\frac{\delta y_3}{\delta e_3} = -\frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1} + \pi_{11}\left[\frac{\delta}{\delta e_1}\pi_{23} - \pi_{23}(\pi_{23} - \pi_{32})\right]$$

$$+ \pi_{22}\left[-\frac{\delta}{\delta e_2}\pi_{13} + \pi_{13}\pi_{31}\right] - \pi_{33}\left[\frac{\delta}{\delta e_3}\pi_{12} + \pi_{12}\pi_{21}\right] + (\pi_{31} + \pi_{13})x_2$$

$$- 2\pi_{12}y_3 + \frac{\alpha_{13}}{2\alpha_{12}}\pi_{13}u_1 - \frac{\alpha_{21}}{2\alpha_{23}}\pi_{12}u_2 - \frac{\alpha_{32}}{2\alpha_{31}}(\pi_{23} - \pi_{32})u_3$$

$$- \pi_{12}\left[(\pi_{13} + \pi_{31})(\pi_{23} + \pi_{32}) - 2\pi_{13}\pi_{23}\right] + \pi_{13}\pi_{21}(\pi_{23} + \pi_{32}) \text{ ,}$$

$$etc. \tag{4.9}$$

From (4.5) and (4.9) we now have, using (1.35),

$$\frac{\delta x_2}{\delta e_2} = -\frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1} + \pi_{11}\left[\frac{\delta}{\delta e_1}\pi_{32} - \pi_{32}(\pi_{23} - \pi_{32})\right] + \pi_{22}\left[-\frac{\delta}{\delta e_2}\pi_{13} + \pi_{13}\pi_{31}\right]$$

$$- \pi_{33}\left[\frac{\delta}{\delta e_3}\pi_{12} + \pi_{12}\pi_{21}\right] + 2\pi_{13}x_2 - (\pi_{12} + \pi_{12})y_3 + \frac{\alpha_{13}}{2\alpha_{12}}\pi_{13}u_1$$

$$- \frac{\alpha_{21}}{2\alpha_{23}}\pi_{12}u_2 - \frac{\alpha_{32}}{2\alpha_{31}}(\pi_{23} - \pi_{32})u_3 - \pi_{12}\left[(\pi_{13} + \pi_{31})(\pi_{23} + \pi_{32})\right.$$

$$\left. - 2\pi_{13}\pi_{23}\right] + \pi_{13}\pi_{21}(\pi_{23} + \pi_{32}) \text{ , etc.} \tag{4.10}$$

The six sets of conditions, taken in pairs (4.1) and (4.4), (4.2) and (4.3), (4.9) and (4.10), give the eighteen gradient components of x_a and y_a. We have obtained these conditions independently by direct computation. The rotation condition serves a check. By applying the transformation given by (1.13), (1.31) and (3.7) to (4.1), (4.2) and (4.9) we obtain (4.4), (4.3) and (4.10) as their respective duals.

5. DERIVATION OF INTEGRALS 1

It appears that while the relatively simple expressions (3.15) and (3.16) are available for the gradients $\frac{\delta u_1}{\delta e_1}$, *etc.*, and $\frac{\delta u_1}{\delta e_3}$, *etc.*, the third set of gradients $\frac{\delta u_1}{\delta e_2}$, *etc.* is much more deeply entrenched. If one looks at (1.29) or (1.30) one sees that the first two sets appear as second order mixed gradients of the abnormalities, while the gradients $\frac{\delta u_1}{\delta e_2}$ depend on terms such as $\frac{\delta^2 \pi_{11}}{\delta e_2^2}$. Our purpose now is to generate expressions for the gradients $\frac{\delta u_1}{\delta e_2}$, *etc.* For purposes of reference we use the term integral to mean a polynomial relation among the eighteen variables π_{ab}, x_a, y_a, u_a and the three gradients $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$.

Substituting directly the expressions from the sets (4.8), (4.9) and (4.10) into the identity

$$\frac{\delta}{\delta e_1}(y_1 + x_1) - \frac{\delta y_1}{\delta e_1} - \frac{\delta x_1}{\delta e_1} = 0$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\alpha_{21}}{\alpha_{23}}\frac{\delta u_2}{\delta e_3} + \frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1} + \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}\frac{\delta u_1}{\delta e_2} - 2\pi_{32}x_1 + 2\pi_{23}y_1 + (\pi_{31} + \pi_{13})(y_2 - x_2) \\
&\quad + (\pi_{12} + \pi_{21})(y_3 - x_3) + \frac{\alpha_{13}}{\alpha_{12}}(\pi_{31} - \pi_{13})u_1 + \frac{\alpha_{21}}{\alpha_{23}}(\pi_{12} - \pi_{21})u_2 \\
&\quad + \frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}(\pi_{23} - \pi_{32})u_3 + 2\pi_{11}(\pi_{23}^2 - \pi_{32}^2) + \pi_{22}(\pi_{31}^2 - \pi_{13}^2) \\
&\quad + \pi_{33}(\pi_{12}^2 - \pi_{21}^2) + \pi_{31}\pi_{12}(2\pi_{23} + \pi_{32})
\end{aligned}$$

$$- \pi_{13} \left[\pi_{32} \, (2\pi_{21} - \pi_{12}) + \pi_{23}(\pi_{12} + \pi_{21})\right] = 0 \, . \tag{5.1}$$

In (5.1) we write

$$(\pi_{31} + \pi_{13})(y_2 - x_2) = - 2\pi_{13}x_2 + 2\pi_{31}y_2 + (\pi_{13} - \pi_{31})(y_2 + x_2) \, ,$$

$$(\pi_{12} + \pi_{21})(y_3 - x_3) = - 2\pi_{21}x_3 + 2\pi_{12}y_3 + (\pi_{21} - \pi_{12})(y_3 + x_3) \, ,$$

and substitute for $y_2 + x_2$ and $y_3 + x_3$ from the system (3.11). We obtain the symmetric relation

$$\begin{aligned}
&\frac{\alpha_{32}}{\alpha_{31}} \left[\frac{\delta u_3}{\delta c_1} + (\pi_{23} - \pi_{32})u_3\right] + \frac{\alpha_{13}}{\alpha_{12}} \left[\frac{\delta u_1}{\delta c_2} + (\pi_{31} - \pi_{13})u_1\right] \\
&\qquad + \frac{\alpha_{21}}{\alpha_{23}} \left[\frac{\delta u_2}{\delta c_3} + (\pi_{12} - \pi_{21})u_2\right] \\
&\qquad - 4 \left[(\pi_{32}x_1 - \pi_{23}y_1) + (\pi_{13}x_2 - \pi_{31}y_2) + (\pi_{21}x_3 - \pi_{12}y_3)\right] \\
&\qquad - 4 \left[\pi_{11}(\pi_{32}^2 - \pi_{23}^2) + \pi_{22}(\pi_{13}^2 - \pi_{31}^2) + \pi_{33}(\pi_{21}^2 - \pi_{12}^2)\right] \\
&\qquad\qquad = 0 \, .
\end{aligned} \tag{5.2}$$

This symmetrical relation is invariant under the rotation given by (1.13), (1.31) and (3.7).

A second integral can be obtained as follows. From (3.8), (1.29), (1.30), (2.2), (3.15) and (3.16) we take the directional derivative of (3.15) with respect to $\underset{\sim}{e}_3$ to obtain

$$\frac{\delta^2 u_1}{\delta e_3 \delta e_1} = -\frac{4\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_3} - 4\frac{\alpha_{21}}{\alpha_{23}}u_2 y_3 + 8\frac{\alpha_{21}}{\alpha_{23}}\pi_{12}\pi_{11}y_3 - \frac{3}{2}\frac{\alpha_{13}}{\alpha_{12}}u_1^2 + 2y_2 u_1$$

$$- \left[-\pi_{22}(3\pi_{13} + 2\pi_{31}) + 3\pi_{21}(\pi_{23} + \pi_{32}) + 2\pi_{32}\pi_{12}\right]u_1$$

$$- (3\pi_{23} - 2\pi_{32})\left[-4\pi_{33}x_1 + (3\pi_{21} - 2\pi_{12})u_1 + \frac{\alpha_{12}}{\alpha_{13}}\pi_{11}u_3\right]$$

$$+ \frac{\alpha_{21}}{\alpha_{23}}\left[2\pi_{33}(\pi_{12} - \pi_{21})u_2 - \pi_{33}\frac{\delta u_2}{\delta e_3}\right]. \tag{5.3}$$

Similarly taking the directional derivative of (3.16) with respect to $\underset{\sim}{e}_1$ we obtain

$$\frac{\delta^2 u_1}{\delta e_1 \delta e_3} = -4\pi_{33}\frac{\delta x_1}{\delta e_1} - 4\frac{\alpha_{12}}{\alpha_{13}}u_3 x_1 - 8\pi_{32}\pi_{33}x_1 - \frac{3}{2}\frac{\alpha_{13}}{\alpha_{12}}u_1^2 + 2x_2 u_1$$

$$+ \left[\pi_{22}(3\pi_{31} + 2\pi_{13}) - 3\pi_{23}(\pi_{12} + \pi_{21}) - 2\pi_{32}\pi_{12}\right]u_1$$

$$+ (3\pi_{21} - 2\pi_{12})\frac{\alpha_{21}}{\alpha_{23}}\left[-4\pi_{11}y_3 - (3\pi_{23} - 2\pi_{32})\frac{\alpha_{23}}{\alpha_{21}}u_1 - \pi_{33}u_2\right]$$

$$+ \frac{\alpha_{12}}{\alpha_{13}}\left[-2\pi_{11}(\pi_{23} - \pi_{32})u_3 + \pi_{11}\frac{\delta u_3}{\delta e_1}\right]. \tag{5.4}$$

From the commutation formula (1.11) with (3.15) and (3.16), we have

$$\frac{\delta^2 u_1}{\delta e_1 \delta e_3} - \frac{\delta^2 u_1}{\delta e_3 \delta e_1} = \frac{\alpha_{21}}{\alpha_{23}} \pi_{12} \left[- 4\pi_{11} y_3 - (3\pi_{23} - 2\pi_{32}) \frac{\alpha_{23}}{\alpha_{21}} u_1 - \pi_{33} u_2 \right]$$

$$+ \pi_{22} \frac{\delta u_1}{\delta e_2}$$

$$+ \pi_{32} \left[- 4\pi_{33} x_1 + (3\pi_{21} - 2\pi_{12}) u_1 \right.$$

$$\left. + \frac{\alpha_{12}}{\alpha_{13}} \pi_{11} u_3 \right] . \qquad (5.5)$$

From (5.3), (5.4) and (5.5) we now obtain

$$- 4\pi_{33} \frac{\delta x_1}{\delta e_1} + 4 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \frac{\delta y_3}{\delta e_3} + \frac{\alpha_{12}}{\alpha_{13}} \pi_{11} \frac{\delta u_3}{\delta e_1} - \pi_{22} \frac{\delta u_1}{\delta e_2} + \frac{\alpha_{21}}{\alpha_{23}} \pi_{33} \frac{\delta u_2}{\delta e_3}$$

$$+ 2(x_2 - y_2) u_1 + 4 \frac{\alpha_{21}}{\alpha_{23}} u_2 y_3 - 4 \frac{\alpha_{12}}{\alpha_{13}} u_3 x_1 + \pi_{22}(\pi_{31} - \pi_{13}) u_1$$

$$+ \frac{\alpha_{21}}{\alpha_{23}} \pi_{33}(\pi_{12} - \pi_{21}) u_2 + \frac{\alpha_{12}}{\alpha_{13}} (\pi_{23} - \pi_{32}) \pi_{11} u_3 - 4(3\pi_{23} - \pi_{32}) \pi_{33} x_1$$

$$- 4 \frac{\alpha_{21}}{\alpha_{23}} (3\pi_{21} - \pi_{12}) \pi_{11} y_3 = 0 , \quad \textit{etc.} \qquad (5.6)$$

On substitution for $\frac{\delta x_1}{\delta e_1}$ and $\frac{\delta y_3}{\delta e_3}$ from the sets (4.9) and (4.10), and substituting for $\frac{\delta}{\delta e_1} \pi_{23}$, $\frac{\delta}{\delta e_1} \pi_{32}$, $\frac{\delta}{\delta e_2} \pi_{13}$ and $\frac{\delta}{\delta e_3} \pi_{12}$ from (1.24), (1.25), (1.26) and (1.27), we obtain the integral

$$3\pi_{33}\frac{\alpha_{21}}{\alpha_{23}}\frac{\delta u_2}{\delta c_3} + 3\pi_{11}\frac{\alpha_{12}}{\alpha_{13}}\frac{\delta u_3}{\delta c_1} - \pi_{22}\frac{\delta u_1}{\delta c_2} + 2(x_2 - y_2)u_1 - 4\frac{\alpha_{12}}{\alpha_{13}}u_3x_1$$

$$+ 4\frac{\alpha_{21}}{\alpha_{23}}u_2y_3 - 4\pi_{33}(\pi_{32} + 3\pi_{23})x_1 + 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}(\pi_{31} + \pi_{13})x_2$$

$$+ 4\pi_{33}(\pi_{31} + \pi_{13})y_2 - 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}(\pi_{12} + 3\pi_{21})y_3$$

$$+ \left[2\left(\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}\pi_{31} - \frac{\alpha_{31}}{\alpha_{32}}\pi_{11}\pi_{13}\right) + \pi_{22}(\pi_{31} - \pi_{13})\right]u_1$$

$$+ \left[3\frac{\alpha_{21}}{\alpha_{23}}\pi_{33}(\pi_{12} - \pi_{21}) - 2\left(\frac{\alpha_{21}}{\alpha_{23}}\right)^2\pi_{11}\pi_{12}\right]u_2$$

$$+ \left[3\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}(\pi_{23} - \pi_{32}) - 2\frac{\alpha_{32}}{\alpha_{31}}\pi_{33}\pi_{32}\right]u_3$$

$$+ 4\pi_{33}\left[\pi_{31}\left[(\pi_{32} + \pi_{23})(\pi_{12} + \pi_{21}) - 2\pi_{32}\pi_{12}\right] - \pi_{32}\pi_{13}(\pi_{12} + \pi_{21})\right]$$

$$- 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left[\pi_{12}\left[(\pi_{13} + \pi_{31})(\pi_{23} + \pi_{32}) - 2\pi_{13}\pi_{23}\right] - \pi_{13}\pi_{21}(\pi_{23} + \pi_{32})\right]$$

$$- 8\pi_{33}^2\pi_{21}^2 + 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}\pi_{12}^2 + 4\left[\pi_{33}^2 - \frac{\alpha_{31}}{\alpha_{13}}\pi_{11}^2 + \frac{\alpha_{31}}{\alpha_{23}}\pi_{11}\pi_{22}\right.$$

$$\left. - \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}\right]\pi_{12}\pi_{21} + 4\pi_{33}\pi_{11}\pi_{32}^2 - 8\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2\pi_{23}^2$$

$$+ 4\left[\frac{\alpha_{12}}{\alpha_{32}}\pi_{33}^2 - \pi_{33}\pi_{11} + \frac{\alpha_{13}}{\alpha_{23}}\pi_{33}\pi_{22} + \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2\right]\pi_{23}\pi_{32}$$

$$- 4 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}\pi_{22}\pi_{13}^2 - 4\pi_{22}\pi_{33}\pi_{31}^2$$

$$+ 4 \left[- \frac{\alpha_{23}}{\alpha_{13}} \pi_{33}\pi_{11} + \pi_{22}\pi_{33} + \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}\pi_{22} - \frac{\alpha_{21}\alpha_{21}}{\alpha_{23}\alpha_{31}} \pi_{11}\pi_{33}\right] \pi_{13}\pi_{31}$$

$$+ \pi_{33} \left\{ \frac{\alpha_{12}}{\alpha_{23}} \pi_{22}^3 + \frac{\alpha_{13}}{\alpha_{32}} \pi_{33}^3 + 4\pi_{11}\pi_{22}\pi_{33} \right.$$

$$+ \left[\frac{\alpha_{23}}{\alpha_{21}} - 2 \frac{\alpha_{12}}{\alpha_{32}} - \frac{\alpha_{13}\alpha_{31}}{\alpha_{21}\alpha_{32}}\right] \pi_{22}\pi_{33}^2 + \left[\frac{\alpha_{32}}{\alpha_{31}} - 2 \frac{\alpha_{13}}{\alpha_{23}} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{31}\alpha_{23}}\right] \pi_{33}\pi_{22}^2$$

$$\left. - 3\left(\frac{\alpha_{21}}{\alpha_{23}} \pi_{22} + \frac{\alpha_{31}}{\alpha_{32}} \pi_{33}\right) \pi_{11}^2 \right\}$$

$$+ \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \left\{ \frac{\alpha_{31}}{\alpha_{12}} \pi_{11}^3 + \frac{\alpha_{32}}{\alpha_{21}} \pi_{22}^3 + 4\pi_{11}\pi_{22}\pi_{33} \right.$$

$$+ \left[\frac{\alpha_{21}}{\alpha_{23}} - 2 \frac{\alpha_{32}}{\alpha_{12}} - \frac{\alpha_{13}\alpha_{31}}{\alpha_{12}\alpha_{23}}\right] \pi_{22}\pi_{11}^2 + \left[\frac{\alpha_{12}}{\alpha_{13}} - 2 \frac{\alpha_{31}}{\alpha_{21}} - \frac{\alpha_{32}\alpha_{23}}{\alpha_{13}\alpha_{21}}\right] \pi_{11}\pi_{22}^2$$

$$\left. - 3\left(\frac{\alpha_{13}}{\alpha_{12}} \pi_{11} + \frac{\alpha_{23}}{\alpha_{21}} \pi_{22}\right) \pi_{33} \right\} + \pi_{11}\pi_{33} \left\{ \frac{\alpha_{21}}{\alpha_{13}} \left(\frac{\alpha_{12}\alpha_{21}}{\alpha_{23}\alpha_{32}} - \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{32}} - 1\right) \pi_{11}^2 \right.$$

$$+ \left(\frac{\alpha_{23}\alpha_{32}}{\alpha_{12}\alpha_{31}} - \frac{\alpha_{13}}{\alpha_{12}} - \frac{\alpha_{21}}{\alpha_{31}}\right) \pi_{33}^2 - 2 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}\pi_{22} - 2\pi_{22}\pi_{33}$$

$$\left. + 2 \frac{\alpha_{23}}{\alpha_{13}} \left(1 - \frac{\alpha_{12}\alpha_{21}}{\alpha_{32}\alpha_{23}}\right) \pi_{33}\pi_{11} \right\} = 0 , \quad etc. \tag{5.7}$$

If one applies the rotation transformation given by (1.13), (1.31) and (3.7) to the condition (5.7) one regains the same condition.

A second check on (5.7) and the relations leading to it is as follows. If one writes (5.7) as

$$- 4 \pi_{33} \frac{\delta x_1}{\delta e_1} + 4\pi_{11} \frac{\alpha_{21}}{\alpha_{23}} \frac{\delta y_3}{\delta e_3} + \frac{\alpha_{12}}{\alpha_{13}} \pi_{11} \frac{\delta u_3}{\delta e_1} - \pi_{22} \frac{\delta u_1}{\delta e_2}$$

$$+ \frac{\alpha_{21}}{\alpha_{23}} \pi_{33} \frac{\delta u_2}{\delta e_3} + \phi_{13} = 0 \qquad (5.8)$$

and eliminates the expression

$$\frac{\alpha_{12}}{\alpha_{13}} \pi_{11} \frac{\delta u_3}{\delta e_1} - \pi_{22} \frac{\delta u_1}{\delta e_2} + \frac{\alpha_{21}}{\alpha_{23}} \pi_{33} \frac{\delta u_2}{\delta e_3}$$

between (5.8) and the condition obtained by permuting the indices forward once, one obtains by (1.22)

$$4 \pi_{11} \left(\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2} \right) + 4 \left(\pi_{22} \frac{\alpha_{32}}{\alpha_{31}} \frac{\delta y_1}{\delta e_1} - \pi_{33} \frac{\alpha_{23}}{\alpha_{21}} \frac{\delta x_1}{\delta e_1} \right)$$

$$+ \phi_{21} + \frac{\alpha_{23}}{\alpha_{21}} \phi_{13} = 0 \ . \qquad (5.9)$$

Since the conditions (5.8) are cyclic, the conditions (5.9) can represent at most only two independent conditions. If one eliminates $\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2}$ from (5.9) by means of (4.5) one obtains the same condition as that obtained by taking the directional derivative with respect to $\underset{\sim}{e}_1$ of the presented member of the set (3.18). This latter condition is only equivalent to two conditions because the three conditons (3.18) are the components of the curl of the vector given by (2.6) so that the divergence must vanish.

6. DERIVATION OF INTEGRALS. 2

Consider the member of the set (3.18)*

$$\pi_{13} \frac{\alpha_{12}}{\alpha_{13}} u_3 + \pi_{23} u_1 + 2\left[\frac{\alpha_{21}}{\alpha_{23}} \pi_{11} y_3 - \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} x_3\right] + 2\pi_{33}^2 \operatorname{div} \underset{\sim}{e}_3 = 0 \;. \quad (6.1)$$

We take the directional derivative of (6.1) with respect to $\underset{\sim}{e}_1$ using (1.35), (3.8), (2.2), (1.29), (1.30) and (3.6). We obtain

$$(y_3 - \pi_{13}\pi_{23} + \pi_{12}\pi_{33}) \frac{\alpha_{12}}{\alpha_{13}} u_3 + \pi_{13} \frac{\alpha_{12}}{\alpha_{13}} \frac{\delta u_3}{\delta e_1} + \left(\frac{\delta}{\delta e_1} \pi_{23}\right) u_1 + \pi_{23} \frac{\delta u_1}{\delta e_1}$$

$$+ 2\left[\frac{\alpha_{21}}{\alpha_{23}} \pi_{32}(2\pi_{11} y_3) + 2\pi_{23}\left(-\frac{\alpha_{21}}{\alpha_{23}} \pi_{11} y_3 + \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} x_3\right) - \frac{\alpha_{12}}{\alpha_{13}} u_3 x_3\right]$$

$$+ 4\pi_{33}\left(\frac{\alpha_{12}}{\alpha_{13}} u_3 + 2\pi_{32}\pi_{33}\right) \operatorname{div} \underset{\sim}{e}_3 + 2\pi_{33}^2 (y_2 + \pi_{32} \operatorname{div} \underset{\sim}{e}_3)$$

$$+ 2\left[\frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \frac{\delta y_3}{\delta e_1} - \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \frac{\delta x_3}{\delta e_1}\right] = 0 \;. \quad (6.2)$$

Eliminating the term $\frac{\alpha_{21}}{\alpha_{23}} \pi_{11} y_3 - \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} x_3$ from (6.2) by means of (6.1) we obtain

*Note that we avoid the operation discussed at the end of Chapter 5.

$$\left[y_3 - 2x_3 + \pi_{13}\pi_{23} + \pi_{33}(5\pi_{12} - 4\pi_{21})\right] \frac{\alpha_{12}}{\alpha_{13}} u_3 + \left[\frac{\delta}{\delta e_1}\pi_{23} + 2\pi_{23}^2\right] u_1$$

$$+ 2\pi_{33}^2 y_2 + 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{32}y_3 + 2\pi_{33}^2 \operatorname{div} \underset{\sim}{e}_3 \; [2\pi_{23} + 5\pi_{32}]$$

$$+ 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_1} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\frac{\delta x_3}{\delta e_1}\right] + \pi_{13}\frac{\alpha_{12}}{\alpha_{13}}\frac{\delta u_3}{\delta e_1}$$

$$+ \pi_{23}\frac{\delta u_1}{\delta e_1} = 0 \;, \; etc. \tag{6.3}$$

Taking the directional derivative of (6.1) with respect to $\underset{\sim}{e}_2$ in a similar manner we get

$$\left[- x_3 + 2y_3 - \pi_{13}\pi_{23} + \pi_{33}\left(4\pi_{12} - 5\pi_{21}\right)\right] u_1$$

$$+ \left[\frac{\delta}{\delta e_2}\pi_{13} - 2\pi_{13}^2\right] \frac{\alpha_{12}}{\alpha_{13}} u_3 + 2\pi_{33}^2 x_1 + 4\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{31}x_3$$

$$- 2\pi_{33}^2 \operatorname{div} \underset{\sim}{e}_3 \; [2\pi_{13} + 5\pi_{31}]$$

$$+ 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_2} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\frac{\delta x_3}{\delta e_2}\right] + \pi_{13}\frac{\alpha_{12}}{\alpha_{13}}\frac{\delta u_3}{\delta e_2}$$

$$+ \pi_{23}\frac{\delta u_1}{\delta e_2} = 0 \;, \; etc. \tag{6.4}$$

The conditions (6.3) and (6.4) have yet to be expressed explicitly in terms of the variables π_{ab}, x_a, y_a, u_a and the three gradients $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$.

The gradients $\frac{\delta u_1}{\delta e_1}$ and $\frac{\delta u_3}{\delta e_2}$ are readily eliminated from (6.3) and (6.4) by substitution from (3.15) and (3.16). Likewise, we eliminate the gradients $\frac{\delta}{\delta e_1}\pi_{23}$ and $\frac{\delta}{\delta e_2}\pi_{13}$ by the relations (1.24) to (1.27). To eliminate $\frac{\delta x_3}{\delta e_1}$, $\frac{\delta y_3}{\delta e_1}$, $\frac{\delta x_3}{\delta e_2}$ and $\frac{\delta y_3}{\delta e_2}$ we use (4.1), (4.2), (4.3) and (4.4). In making this substitution expressions for $\frac{\delta}{\delta e_1}\pi_{23}$, $\frac{\delta}{\delta e_1}\pi_{32}$, $\frac{\delta}{\delta e_2}\pi_{31}$, $\frac{\delta}{\delta e_2}\pi_{13}$ are obtained directly from (1.24) to (1.27). Expressions for the second order mixed derivatives in (4.1) to (4.4) are naturally more complicated, but they are all obtainable by taking the appropriate gradients of the relations (1.24) to (1.27). We illustrate by considering $\frac{\delta^2}{\delta e_2 \delta e_1}\pi_{23}$ occurring in the expression (4.1) for $\frac{\delta x_3}{\delta e_1}$. We have

$$\frac{\delta^2}{\delta e_2 \delta e_1}\pi_{23} = -2\pi_{23}\frac{\delta}{\delta e_2}\pi_{23} + \frac{\alpha_{32}}{\alpha_{12}}\left(\pi_{21}\frac{\delta}{\delta e_2}\pi_{12} + \pi_{12}\frac{\delta}{\delta e_2}\pi_{21}\right)$$

$$+ \frac{\partial\eta_{123}}{\partial\pi_{11}}\frac{\delta\pi_{11}}{\delta e_2} + \frac{\partial\eta_{123}}{\partial\pi_{22}}\frac{\delta\pi_{22}}{\delta e_2} + \frac{\partial\eta_{123}}{\partial\pi_{33}}\frac{\delta\pi_{33}}{\delta e_2}, \qquad (6.5)$$

where $\frac{\delta}{\delta e_2}\pi_{23}$, $\frac{\delta}{\delta e_2}\pi_{12}$, $\frac{\delta}{\delta e_2}\pi_{21}$, $\frac{\delta\pi_{11}}{\delta e_2}$, $\frac{\delta\pi_{22}}{\delta e_2}$, $\frac{\delta\pi_{33}}{\delta e_2}$ are given respectively by (3.9), (1.32), (3.8), (1.29), (2.2) and (1.30), while by (1.27)

$$\frac{\partial\eta_{123}}{\partial\pi_{11}} = \frac{1}{2}\left(-\frac{\alpha_{32}}{\alpha_{12}}\pi_{22} + \frac{\alpha_{31}}{\alpha_{12}}\pi_{11}\right) , \tag{6.6}$$

$$\frac{\partial\eta_{123}}{\partial\pi_{22}} = \frac{1}{2}\left(\pi_{33} - \frac{\alpha_{32}}{\alpha_{12}}\pi_{11} + \frac{\alpha_{32}}{\alpha_{31}}\left(\frac{\alpha_{32}}{\alpha_{12}} - \frac{\alpha_{21}}{\alpha_{23}}\right)\pi_{22}\right), \tag{6.7}$$

$$\frac{\partial\eta_{123}}{\partial\pi_{33}} = \frac{1}{2}\left(\pi_{22} - 3\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}\right) . \tag{6.8}$$

After a somewhat tedious but straight-forward calculation we obtain from (6.3) and (6.4) respectively the two sets of integrals

$$\frac{\alpha_{12}}{\alpha_{13}}\pi_{13}\frac{\delta u_3}{\delta e_1} + \frac{\alpha_{12}}{\alpha_{13}}(y_3 - 2x_3)u_3 + 2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}(3\pi_{32} - 2\pi_{23})y_3$$

$$+ 2\left(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right)\left(2\pi_{23}x_3 + \frac{\alpha_{32}}{\alpha_{12}}\pi_{12}y_1\right)$$

$$+ 2\pi_{33}\left(\pi_{33}y_2 - \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}x_2\right) + \left\{\frac{5}{4}\frac{\alpha_{31}}{\alpha_{12}}\pi_{11}^2 + \frac{5}{4}\left(\frac{\alpha_{12}}{\alpha_{13}} - \frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{13}}\right)\pi_{22}^2\right.$$

$$- \frac{3}{4}\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}^2 + \left[2\frac{\alpha_{21}}{\alpha_{23}} - \frac{5}{2}\frac{\alpha_{32}}{\alpha_{12}}\right]\pi_{11}\pi_{22} + 2\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}\pi_{33} - \frac{9}{2}\pi_{22}\pi_{33}$$

$$+ 2\pi_{23}\pi_{32} + \frac{\alpha_{32}}{\alpha_{12}}\pi_{12}\pi_{21} \Bigg\} u_1 + \frac{\alpha_{21}}{\alpha_{23}} \left[2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{32} - \left(\frac{\alpha_{12}}{\alpha_{13}}\pi_{22} + \pi_{33}\right)\pi_{23} \right] u_2$$

$$+ \Bigg\{ \left(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right) \frac{\alpha_{32}\alpha_{32}}{\alpha_{12}\alpha_{31}}\pi_{21} + \frac{\alpha_{12}}{\alpha_{13}} \left(3\pi_{13}\pi_{23} + \pi_{33}(5\pi_{12} - 4\pi_{21})\right)$$

$$+ 2\left(\frac{\alpha_{12}}{\alpha_{13}}\right)^2 \pi_{22}\pi_{21} \Bigg\} \quad u_3 + 2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} \Bigg\{ - \frac{\alpha_{32}}{\alpha_{12}} \left[(\pi_{12} - \pi_{21})\pi_{13} + \pi_{12}\pi_{31}\right]\pi_{21}$$

$$- \frac{\alpha_{23}}{\alpha_{13}}\pi_{13}\pi_{31}(\pi_{13} + \pi_{31}) + (\pi_{23}\pi_{32} + 2\pi_{32}^2)\pi_{13}$$

$$+ \left[2\pi_{32}(\pi_{23} + \pi_{32}) + \frac{\alpha_{32}}{\alpha_{12}}\pi_{12}\pi_{21}\right]\pi_{31} \Bigg\}$$

$$- 2\frac{\alpha_{12}}{\alpha_{13}}\pi_{22} \Bigg\{ \frac{\alpha_{32}}{\alpha_{12}} \left[(\pi_{12} - \pi_{21})\pi_{13} + \pi_{12}\pi_{31}\right]\pi_{21}$$

$$+ \pi_{23}\left[(\pi_{23} + \pi_{32})\pi_{31} + \pi_{32}\pi_{13}\right] \Bigg\}$$

$$+ \left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right] \left[4\pi_{33}\pi_{23}\pi_{21} + 2\frac{\alpha_{32}}{\alpha_{12}}\pi_{11}(\pi_{12}\pi_{23} - \pi_{21}\pi_{32})\right]$$

$$- 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}\pi_{12}\pi_{32} + 2\pi_{33}^2(4\pi_{23} + 5\pi_{32})(\pi_{12} - \pi_{21})$$

$$- 2\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{33}(\pi_{23}\pi_{12} - \pi_{21}\pi_{32})$$

$$+ \left\{ \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \left[\left(\frac{\alpha_{31}}{\alpha_{12}} - \frac{1}{2} \frac{\alpha_{21}}{\alpha_{13}} \right) \pi_{11}^2 - \left(\frac{3}{2} \frac{\alpha_{12}}{\alpha_{13}} + \frac{\alpha_{32}\alpha_{23}}{\alpha_{21}\alpha_{13}} \right) \pi_{22}^2 \right. \right.$$

$$\left. + \left(\frac{5}{2} \frac{\alpha_{13}}{\alpha_{12}} + \frac{1}{2} \frac{\alpha_{23}\alpha_{32}}{\alpha_{12}\alpha_{31}} \right) \pi_{33}^2 - 2 \frac{\alpha_{32}}{\alpha_{12}} \pi_{11}\pi_{22} - \pi_{22}\pi_{33} + \frac{\alpha_{23}}{\alpha_{13}} \pi_{33}\pi_{11} \right]$$

$$+ \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \left[\frac{\alpha_{31}}{\alpha_{12}} \pi_{11}^2 + \left(\frac{\alpha_{12}}{\alpha_{13}} - \frac{\alpha_{32}\alpha_{23}}{\alpha_{21}\alpha_{13}} \right) \pi_{22}^2 + 3 \frac{\alpha_{13}}{\alpha_{12}} \pi_{33}^2 \right.$$

$$\left. \left. - 2 \frac{\alpha_{32}}{\alpha_{12}} \pi_{11}\pi_{22} + 2\pi_{22}\pi_{33} \right] \right\} \pi_{13}$$

$$+ \left\{ \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \left[\frac{1}{2} \left(\frac{\alpha_{31}}{\alpha_{12}} - \frac{\alpha_{21}}{\alpha_{13}} \right) \pi_{11}^2 + \left(4 \frac{\alpha_{12}}{\alpha_{13}} + \frac{3}{2} \frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{13}} \right) \pi_{22}^2 \right. \right.$$

$$\left. + \left(4 \frac{\alpha_{13}}{\alpha_{12}} + \frac{1}{2} \frac{\alpha_{23}\alpha_{32}}{\alpha_{12}\alpha_{31}} \right) \pi_{33}^2 + \frac{\alpha_{32}}{\alpha_{12}} \pi_{11}\pi_{22} - 8\pi_{22}\pi_{33} + \frac{\alpha_{23}}{\alpha_{13}} \pi_{33}\pi_{11} \right]$$

$$+ \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \left[2 \left(- \frac{\alpha_{12}}{\alpha_{13}} + \frac{\alpha_{32}\alpha_{23}}{\alpha_{21}\alpha_{13}} \right) \pi_{22}^2 + 6 \frac{\alpha_{13}}{\alpha_{12}} \pi_{33}^2 + 2 \frac{\alpha_{32}}{\alpha_{12}} \pi_{11}\pi_{22} \right.$$

$$\left. \left. - 2\pi_{22}\pi_{33} \right] \right\} \pi_{31} = 0 \ , \ etc. \tag{6.9}$$

$$\pi_{23} \frac{\delta u_1}{\delta e_2} + (-x_3 + 2y_3)u_1 + 2 \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \ (3\pi_{31} - 2\pi_{13})x_3$$

$$+ 2\left(\frac{\alpha_{21}}{\alpha_{23}} \pi_{11} + \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \right) \left(2\pi_{13}y_3 + \frac{\alpha_{31}}{\alpha_{21}} \pi_{21}x_2 \right)$$

$$+ 2\pi_{33}\left(\pi_{33}x_1 - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}y_1\right) + \frac{\alpha_{23}}{\alpha_{21}}\left[-\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right]\frac{\alpha_{31}\alpha_{31}}{\alpha_{32}\alpha_{21}}\pi_{12}\right.$$

$$\left. - \frac{\alpha_{21}}{\alpha_{23}}\left[3\pi_{23}\pi_{13} + \pi_{33}(5\pi_{21} - 4\pi_{12})\right] - 2\left(\frac{\alpha_{21}}{\alpha_{23}}\right)^2 \pi_{11}\pi_{12}\right] u_1$$

$$+ \frac{\alpha_{21}}{\alpha_{23}}\left[2\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{31} - \left(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} + \pi_{33}\right)\pi_{13}\right] u_2$$

$$+ \frac{\alpha_{12}}{\alpha_{13}}\left[-\frac{5}{4}\frac{\alpha_{32}}{\alpha_{21}}\pi_{22}^2 + \frac{3}{4}\frac{\alpha_{23}}{\alpha_{21}}\pi_{33}^2 - \frac{5}{4}\left(\frac{\alpha_{21}}{\alpha_{23}} - \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}}\right)\pi_{11}^2\right.$$

$$- 2\frac{\alpha_{32}}{\alpha_{31}}\pi_{22}\pi_{33} + \frac{9}{2}\pi_{33}\pi_{11} - \left(2\frac{\alpha_{12}}{\alpha_{13}} - \frac{5}{2}\frac{\alpha_{31}}{\alpha_{21}}\right)\pi_{11}\pi_{22} - \frac{\alpha_{31}}{\alpha_{21}}\pi_{21}\pi_{12}$$

$$\left. - 2\pi_{31}\pi_{13}\right] u_3 + 2\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\left[-\frac{\alpha_{31}}{\alpha_{21}}\left[(\pi_{21} - \pi_{12})\pi_{23} + \pi_{21}\pi_{32}\right]\pi_{12}\right.$$

$$- \frac{\alpha_{13}}{\alpha_{23}}\pi_{23}\pi_{32}(\pi_{23} + \pi_{32}) + (\pi_{13}\pi_{31} + 2\pi_{31}^2)\pi_{23} + \left(2\pi_{31}(\pi_{31} + \pi_{13})\right.$$

$$\left.\left. + \frac{\alpha_{31}}{\alpha_{21}}\pi_{21}\pi_{12}\right)\pi_{32}\right] - 2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left[\frac{\alpha_{31}}{\alpha_{21}}\left[(\pi_{21} - \pi_{12})\pi_{23} + \pi_{21}\pi_{32}\right]\pi_{12}\right.$$

$$\left.\left. + \pi_{13}\left[(\pi_{31} + \pi_{13})\pi_{32} + \pi_{31}\pi_{23}\right]\right]\right]$$

$$+ \left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right]\left[4\pi_{33}\pi_{13}\pi_{12} + 2\frac{\alpha_{31}}{\alpha_{21}}\pi_{22}(\pi_{21}\pi_{13} - \pi_{12}\pi_{31})\right]$$

$$- 4 \frac{\alpha_{12}}{\alpha_{13}} \pi_{22}\pi_{33}\pi_{31}\pi_{21} + 2\pi_{33}^{2}(4\pi_{13} + 5\pi_{31})(\pi_{21} - \pi_{12})$$

$$- 2 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}\pi_{33} (\pi_{13}\pi_{21} - \pi_{31}\pi_{12})$$

$$+ \left\{ \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \left[\left(\frac{\alpha_{32}}{\alpha_{21}} - \frac{1}{2} \frac{\alpha_{12}}{\alpha_{23}} \right) \pi_{22}^{2} + \left(\frac{5}{2} \frac{\alpha_{23}}{\alpha_{21}} + \frac{1}{2} \frac{\alpha_{31}\alpha_{13}}{\alpha_{21}\alpha_{32}} \right) \pi_{33}^{2} \right.\right.$$

$$\left. - \left(\frac{3}{2} \frac{\alpha_{21}}{\alpha_{23}} + \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}} \right) \pi_{11}^{2} + \frac{\alpha_{13}}{\alpha_{23}} \pi_{22}\pi_{33} - \pi_{33}\pi_{11} - 2 \frac{\alpha_{31}}{\alpha_{21}} \pi_{11}\pi_{22} \right]$$

$$+ \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \left[\frac{\alpha_{32}}{\alpha_{21}} \pi_{22}^{2} + 3 \frac{\alpha_{23}}{\alpha_{21}} \pi_{33}^{2} + \left(\frac{\alpha_{21}}{\alpha_{23}} - \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}} \right) \pi_{11}^{2} + 2\pi_{33}\pi_{11} \right.$$

$$\left.\left. - 2 \frac{\alpha_{31}}{\alpha_{21}} \pi_{11}\pi_{22} \right] \right\} \pi_{23}$$

$$+ \left\{ \frac{\alpha_{12}}{\alpha_{13}} \pi_{22} \left[\frac{1}{2} \left(\frac{\alpha_{32}}{\alpha_{21}} - \frac{\alpha_{12}}{\alpha_{23}} \right) \pi_{22}^{2} + \left(4 \frac{\alpha_{23}}{\alpha_{21}} + \frac{1}{2} \frac{\alpha_{31}\alpha_{13}}{\alpha_{21}\alpha_{32}} \right) \pi_{33}^{2} \right.\right.$$

$$\left. + \left(4 \frac{\alpha_{21}}{\alpha_{23}} + \frac{3}{2} \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}} \right) \pi_{11}^{2} + \frac{\alpha_{13}}{\alpha_{23}} \pi_{22}\pi_{33} - 8\pi_{33}\pi_{11} + \frac{\alpha_{31}}{\alpha_{21}} \pi_{11}\pi_{22} \right]$$

$$+ \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \left[6 \frac{\alpha_{23}}{\alpha_{21}} \pi_{33}^{2} + 2 \left(- \frac{\alpha_{21}}{\alpha_{23}} + \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}} \right) \pi_{11}^{2} - 2\pi_{33}\pi_{11} \right.$$

$$\left.\left. + 2 \frac{\alpha_{31}}{\alpha_{21}} \pi_{11}\pi_{22} \right] \right\} \pi_{32} = 0 , \quad etc. \tag{6.10}$$

While the conditions (6.9) and (6.10) were worked out independently, it may easily be verified that they are duals. One may obtain (6.10) by applying the rotation transformation given by (1.13), (1.31) and (3.7) to the condition (6.9).

PARENTHESIS

No simple new conditions are obtainable by applying the commutation formulae (1.11) to the curvatures and abnormalities. The evaluation of expressions such as

$$\left(\frac{\delta^2}{\delta e_3 \delta e_2 \delta e_1} - \frac{\delta^2}{\delta e_2 \delta e_3 \delta e_1}\right) \pi_{23}$$

gives identities.

It was hoped that another set of integrals of the same order as (5.7), (6.9) and (6.10) could be obtained by taking the directional derivative of the symmetrical condition (5.2). The procedure is as follows.

Taking the gradient of (5.2) with respect to $\underset{\sim}{e}_1$ we obtain the second gradients $\frac{\delta^2 u_3}{\delta e_1^2}$, $\frac{\delta^2 u_1}{\delta e_1 \delta e_2}$, $\frac{\delta^2 u_2}{\delta e_1 \delta e_3}$. The two mixed gradients are reduced by the commutation formula (1.11). The gradient $\frac{\delta u_3^2}{\delta e_1^2}$ can be expressed in terms of $\frac{\delta^2 x_2}{\delta e_1 \delta e_2}$ by (4.10), and hence, using (1.11) in terms of $\frac{\delta^2}{\delta e_2 \delta e_3 \delta e_1} (\pi_{23} - \pi_{32})$ by (4.3). This term can eventually be reduced. Alternatively, $\frac{\delta^2 u_3}{\delta e_1^2}$ can be expressed in terms of $\frac{\delta^2 y_3}{\delta e_1 \delta e_3}$ by (4.9), and a similar, dual procedure adopted. Before realizing that this opening of two paths strongly indicated that the calculations would lead to an identity, we carried out the procedure completely and indeed obtained an identity. The reduction requires the expressions (6.9) and (6.10) and gives a verification of their correctness.

7. PROOF OF MAIN THEOREM 1.1

With the six quantities x_a and y_a given in terms of u_1, u_2 and u_3 by (3.19), we have ten algebraic relations among the variables $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$, u_1, u_2, u_3, the six curvatures π_{ab}, the three abnormalities π_{aa} and the two independent ratios $\frac{\sigma_1^2}{\sigma_2^2}$, $\frac{\sigma_2^2}{\sigma_3^2}$. These ten relations are the symmetrical condition (5.2), the three conditions (5.7) and the six conditions given by (6.9) and (6.10).

If these conditions are independent at each level of the elimination we can in principle eliminate $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$, u_1, u_2, u_3 to obtain four homogeneous polynomials

$$F_\alpha\left(\pi_{aa}, \pi_{ab}, \frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}\right) = 0 , \qquad \alpha = 1 \ldots 4 . \tag{7.1}$$

We are then faced with the problem of determining whether the conditions (7.1) are independent. For example, they could possess a symmetrical common factor. It may be possible to resolve these immense difficulties using computer symbolic mathematics systems (1979[1])*.

We can, however, prove that there exist at least *one* homogeneous polynomial of the type (7.1). We do this by substituting appropriate values for π_{aa}, π_{ab}, σ_1^2, σ_2^2, σ_3^2, performing the elimination, and showing that the eliminant does not vanish for these values. In choosing the

*We quote a statement given in this reference: "Today there are computer symbolic mathematics systems, which can, among other things, quickly and exactly expand or factor complicated expressions having hundreds or thousands of terms."

values we must be careful that the order of the equations is not reduced at any stage of the elimination.

We choose $\pi_{11} = 1$, $\pi_{22} = 2$, $\pi_{33} = -1$, and all $\pi_{ab} = 1$, $a \neq b$. We take

$$\sigma_1^2 : \sigma_2^2 : \sigma_3^2 = 1 : 2 : 3 ,^*$$

so that

$$\alpha_{12} = \frac{1}{2} , \quad \alpha_{22} = \frac{1}{3} , \quad \alpha_{31} = -2$$

$$\alpha_{21} = -1 , \alpha_{32} = -\frac{1}{2}, \alpha_{13} = \frac{2}{3} . \tag{7.2}$$

From (3.19) we have

$$x_1 = \frac{1}{5} \left[-u_1 + 3u_2 + \frac{3}{8} u_3 + 3 \right] ,$$

$$y_1 = -\frac{1}{5} \left[-u_1 + 3u_2 - \frac{u_3}{4} - 2 \right] ,$$

$$x_2 = \frac{1}{8} \left[-\frac{8}{3} u_1 + 6u_2 + \frac{3}{2} u_3 + 4 \right] , \tag{7.3}$$

$$y_2 = -\frac{1}{8} \left[-8u_1 + 6u_2 + \frac{3}{2} u_3 + 12 \right] ,$$

$$x_3 = -\frac{u_1}{3} - 3u_2 - \frac{u_3}{4} + 10 ,$$

$$y_3 = \frac{u_1}{3} + \frac{3}{2} u_2 + \frac{u_3}{4} - 5 .$$

*In accordance with (1.5) this means

The conditions (6.9) take the form

$$\frac{3}{4}\frac{\delta u_3}{\delta e_1} + \frac{9}{16}u_3^2 + \frac{3}{4}u_3u_1 + \frac{45}{8}u_3u_2 - \frac{39}{10}u_1 + \frac{207}{10}u_2 - \frac{639}{40}u_3 - \frac{839}{5} = 0 ,$$

$$-\frac{1}{3}\frac{\delta u_1}{\delta e_2} - \frac{1}{5}u_1^2 + \frac{3}{5}u_1u_2 + \frac{1}{30}u_1u_3 + \frac{24}{5}u_1 + \frac{571}{15}u_2 + \frac{41}{120}u_3 - \frac{2719}{90} = 0 ,$$

$$4\frac{\delta u_2}{\delta e_3} - 9u_2^2 - \frac{9}{4}u_2u_3 + \frac{20}{3}u_1u_2 - \frac{736}{45}u_1 + \frac{252}{5}u_2 - \frac{9}{20}u_3 - \frac{122}{45} = 0 , \quad (7.4)$$

while from (6.10) we have

$$\frac{\delta u_1}{\delta e_2} + u_1^2 + 6u_1u_2 + \frac{3}{4}u_1u_3 - \frac{104}{5}u_1 - \frac{441}{10}u_2 - \frac{51}{5}u_3 + \frac{949}{10} = 0 ,$$

$$\frac{\delta u_2}{\delta e_3} - \frac{9}{5}u_2^2 + \frac{3}{5}u_1u_2 + \frac{1}{40}u_2u_3 - \frac{667}{90}u_1 + \frac{31}{10}u_2 + \frac{37}{40}u_3 + \frac{553}{45} = 0 ,$$

$$\frac{\delta u_3}{\delta e_1} - \frac{9}{16}u_3^2 + \frac{7}{3}u_1u_3 - \frac{9}{4}u_2u_3 + \frac{416}{45}u_1 - \frac{131}{15}u_2 - \frac{59}{5}u_3 + \frac{15682}{45} = 0 . \quad (7.5)$$

Eliminating the gradients $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$ from (7.4) and (7.5) we get

$$u_1^2 = -\frac{39}{2}u_1u_2 - \frac{17}{8}u_1u_3 + 16u_1 - \frac{701}{4}u_2 + \frac{367}{16}u_3 - \frac{32}{3} ,$$

$$u_2^2 = -\frac{47}{36}u_2u_3 + \frac{64}{27}u_1u_2 + \frac{598}{81}u_1 + \frac{190}{9}u_2 - \frac{83}{36}u_3 - \frac{778}{27} , \quad (7.6)$$

$$u_3^2 = \frac{64}{63}u_1u_3 - \frac{52}{7}u_2u_3 + \frac{2080}{189}u_1 - \frac{1744}{63}u_2 + \frac{152}{21}u_3 + \frac{82400}{189} .$$

The conditions (5.7) become

$$9\,\frac{\delta u_2}{\delta e_3} + \frac{9}{4}\,\frac{\delta u_3}{\delta e_1} - 2\,\frac{\delta u_1}{\delta e_2} - \frac{8}{3}\,u_1^2 - 18\,u_2^2 - \frac{9}{40}\,u_3^2 - u_1u_2 - \frac{24}{5}\,u_2u_3$$

$$+ \frac{27}{20}\,u_3u_1 + \frac{92}{15}\,u_1 + \frac{558}{5}\,u_2 + \frac{89}{10}\,u_3 - \frac{5096}{15} = 0\ ,$$

$$\frac{3}{4}\,\frac{\delta u_3}{\delta e_1} - 2\,\frac{\delta u_1}{\delta e_2} + \frac{\delta u_2}{\delta e_3} - \frac{4}{9}\,u_1^2 - 9\,u_2^2 + \frac{1}{20}\,u_3^2 - \frac{1}{3}\,u_1u_2 - \frac{8}{5}\,u_2u_3$$

$$+ \frac{9}{20}\,u_3u_1 + \frac{46}{15}\,u_1 + \frac{69}{5}\,u_2 - \frac{9}{4}\,u_3 - \frac{28}{15} = 0\ , \qquad (7.7)$$

$$8\,\frac{\delta u_1}{\delta e_2} - 12\,\frac{\delta u_2}{\delta e_3} - \frac{\delta u_3}{\delta e_1} + \frac{16}{3}\,u_1^2 + 48\,u_2^2 + \frac{1}{20}\,u_3^2 + \frac{4}{3}\,u_1u_2 + \frac{32}{5}\,u_2u_3$$

$$- \frac{9}{5}\,u_3u_1 + \frac{296}{9}\,u_1 - 84\,u_2 + \frac{71}{15}\,u_3 - \frac{2720}{9} = 0\ .$$

Using (7.5) to eliminate $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$ from (7.7), and then using (7.6) to eliminate u_1^2, u_2^2, u_3^2, we obtain the three relations:

$$\frac{43}{5}\,u_1u_2 - \frac{187}{35}\,u_2u_3 + \frac{31}{12(35)}\,u_3u_1 - \frac{652}{315}\,u_1 + \frac{13687}{210}\,u_2 + \frac{24919}{840}\,u_3$$

$$- \frac{167569}{315} = 0\ ,$$

$$- \frac{109}{3} u_1u_2 + \frac{417}{70} u_2u_3 - \frac{1103}{420} u_3u_1 - \frac{57767}{945} u_1 - \frac{106811}{210} u_2 + \frac{136231}{2520} u_3$$

$$+ \frac{97927}{315} = 0 ,$$

$$\frac{676}{9} u_1u_2 - \frac{2752}{105} u_2u_3 - \frac{101}{315} u_3u_1 + \frac{83852}{315} u_1 + \frac{420878}{315} u_2 - \frac{2015}{42} u_3$$

$$- \frac{343244}{189} = 0 . \tag{7.8}$$

The single symmetrical condition (5.2)* becomes

$$- \frac{1}{8} \frac{\delta u_3}{\delta e_1} - \frac{2}{3} \frac{\delta u_1}{\delta e_2} + \frac{3}{2} \frac{\delta u_2}{\delta e_3} - \frac{24}{5} u_1 - \frac{18}{5} u_2 - \frac{1}{5} u_3 + \frac{172}{5} = 0 . \tag{7.9}$$

Eliminating the gradients $\frac{\delta u_3}{\delta e_1}$, $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$ from (7.9) by (7.5) and then eliminating u_1^2 , u_2^2 , u_3^2 by (7.6), we obtain

$$- \frac{7}{2} u_1u_2 - \frac{93}{28} u_2u_3 - \frac{39}{56} u_3u_1 + \frac{7381}{315} u_1 - \frac{3382}{35} u_2 - \frac{137}{105} u_3$$

$$+ \frac{253}{35} = 0 . \tag{7.10}$$

*The condition (5.2) is divided by 2.

Eliminating u_1u_2 from (7.10) and (7.8^1), (7.8^1) and (7.8^2), and (7.8^2) and (7.8^3) respectively and dividing through by the coefficient of u_1u_3 in each case, we obtain

$$- u_1u_3 - 8.2471987\ u_2u_3 + 33.898145\ u_1 - 105.1988\ u_2 + 16.159313\ u_3 - 314.03381 = 0\ ,$$

$$- u_1u_3 - 7.1792979\ u_2u_3 - 30.191441\ u_1 - 100.79129\ u_2 + 77.512155\ u_3 - 836.7637 = 0\ ,$$

$$- u_1u_3 - 2.4165543\ u_2u_3 + 24.318844\ u_1 + 49.50761\ u_2 + 11.092884\ u_3 - 204.08588 = 0\ . \qquad (7.11)$$

Subtracting $(7.11)^1$ and $(7.11)^2$, and then $(7.11)^2$ and $(7.11)^3$ and dividing by the coefficient of u_2u_3 we get

$$- u_2u_3 + 60.01455\ u_1 - 4.1272653\ u_2 - 57.451817\ u_3 + 489.49292 = 0\ ,$$

$$- u_2u_3 - 11.445143\ u_1 - 31.55721\ u_2 + 13.94559\ u_3 - 132.83894 = 0\ . \qquad (7.12)$$

Eliminating u_2u_3 from the equations (7.12) we get

$$u_1 = - 0.3838519\ u_2 + 0.9991283\ u_3 - 8.7088515\ . \qquad (7.13)$$

Substituting the expression for u_1 given by (7.13) into (7.11)[1] and (7.12) respectively, we get

$$- 7.8633468\ u_2u_3 - 0.9991283\ u_3^2 - 118.21066\ u_2 + 58.736759\ u_3$$
$$- 609.24772 = 0\ , \tag{7.14}$$

and

$$- u_2u_3 - 27.163964\ u_2 + 2.510418\ u_3 - 33.16488 = 0\ . \tag{7.15}$$

Solving (7.14) and (7.15) for u_2 and u_2u_3 in terms of u_3, we get

$$u_2 = 0.0104742\ u_3^2 - 0.4088151\ u_3 + 3.6530498 \tag{7.16}$$

$$u_2u_3 = -\ 0.2845213\ u_3^2 + 13.615457\ u_3 - 132.39618\ . \tag{7.17}$$

Equations (7.16) and (7.17) can be reduced to a cubic in u_3. However, a simpler eliminant is obtained if we reduce the equations to quadratics in u_3.

Eliminating u_1 between (7.10) and (7.13) we obtain

$$u_2^2 - 4.8761754\ u_2u_3 - 0.5179237\ u_3^2 - 55.93073\ u_2 + 20.969134\ u_3$$
$$- 146.51113 = 0\ . \tag{7.18}$$

From (7.13) we have

$$u_1^2 = 0.1473422\ u_2^2 + 0.9982573\ u_3^2 - 0.7670345\ u_2u_3 + 6.6858183\ u_2$$

$$- 17.402519\ u_3 + 75.844094\ . \tag{7.19}$$

Using (7.13) and (7.19) to eliminate u_1 from the equations (7.6) we get

$$- u_2^2 + 0.4253887\ u_3^2 + 2.4394717\ u_2u_3 + 2.4877913\ u_2 - 10.198245\ u_3$$

$$+ 30.779431 = 0\ , \tag{7.20}$$

$$u_2^2 - 0.5564504\ u_2u_3 + 1.2388066\ u_2 - 2.6550089\ u_3$$

$$+ 48.751871 = 0\ , \tag{7.21}$$

$$- 0.0149874\ u_3^2 + 7.8185142\ u_2u_3 + 31.90694\ u_2 - 9.386705\ u_3$$

$$- 340.13539 = 0\ . \tag{7.22}$$

We note that (7.22) does not involve u_2^2, this is apparent from the form of (7.6^3).

Eliminating u_2^2 between (7.18) and (7.20) and between (7.18) and (7.21) we now obtain

$$- 2.4367037\ u_2u_3 - 0.092535\ u_3^2 - 53.442939\ u_2 + 10.770889\ u_3$$

$$- 115.7317 = 0\ , \tag{7.23}$$

and

$$4.319725\ u_2u_3 + 0.5179237\ u_3^2 + 57.169536\ u_2 - 23.624142\ u_3$$

$$+ 195.263 = 0\ . \tag{7.24}$$

We now use (7.16) and (7.17) to eliminate u_2 and u_2u_3 from (7.23) and (7.24). We obtain the three quadratic equations in u_3

$$u_3^2 - 44.098542\ u_3 + 660.6334 = 0\ , \tag{7.25}$$

$$u_3^2 - 13.605866\ u_3 + 284.20771 = 0\ , \tag{7.26}$$

$$u_3^2 - 105.22259\ u_3 + 1493.9601 = 0\ . \tag{7.27}$$

The eliminant of x between the two quadratic equations

$$ax^2 + bx + c = 0\ , \qquad a^1x^2 + b^1x + c^1 = 0\ ,$$

is

$$E = (ca^1 - c^1a)^2 - (bc^1 - b^1c)(ab^1 - a^1b)\ ,$$

and the vanishing of E is the condition that the equations in question have a common root.

The eliminants of u_3 between (7.25) and (7.26) and between (7.26) and (7.27) are respectively

$$E_1 = 249782.25 \quad \text{and} \quad E_2 = 2341047.1 \ .$$

Setting E_1 and E_2 to be zero we have the numerical representations of two relations of the type (7.1). However, this numerical analysis cannot tell us whether these relations are independent.*

We can guarantee the existence of one relation (7.1) but that is all. The relation may be one of three relations connected by cyclic permutation and by the rotation condition, or it may be a single symmetric condition invariant under the rotation transformation. In either case we can assert the existence of one symmetric condition invariant under the rotation and reflexion transformations.* This proves Main Theorem I.1.

*A third such eliminant could be obtained from the cubic represented by (7.16) and (7.17) and one of the quadratics.

8. PROOF OF MAIN THEOREM 1.2

It may happen that the relation whose existence was guaranteed in the previous chapter, is of the form

$$\phi\left(\frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}\right) G\left(\pi_{aa}, \pi_{ab}, \frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}\right) = 0 \quad . \tag{8.1}$$

In this case the condition would be satisfied by the relation among the proper numbers

$$\phi\left(\frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}\right) = 0 \ . \tag{8.2}$$

We cannot discount the possibility that a new class of solutions may indeed be characterized by a functional relation among the proper numbers.*

We postulate that the function ϕ is *not* of the form

$$\frac{\dfrac{\sigma_3^2}{\sigma_1^2} - 1}{\dfrac{\sigma_2^2}{\sigma_1^2} - 1} - k_n \frac{\sigma_3^2}{\sigma_2^2} \tag{8.3}$$

*The function ϕ need not be symmetric where k_n is a constant, and that it does not contain (8.3) as a factor. Evidently due to the finite degree of (8.1) there can be at most a limited number of possible constants k_n. We note that $k_n = 0$ gives the degenerate and excluded case $\sigma_1^2 = \sigma_3^2$. When the expression (8.3) is zero we have

$$\frac{\alpha_{13}}{\alpha_{12}} - k_n = 0 \quad , \tag{8.4}$$

which by the identity (1.23), is equivalent to

$$\frac{\alpha_{21}}{\alpha_{23}} - \frac{1}{1-k_n} = 0 \ , \tag{8.5}$$

$$\frac{\alpha_{32}}{\alpha_{31}} - \frac{k_n-1}{k_n} = 0 \ .$$

This case has to be excluded from the argument. It remains as a possibility.

By (8.2) and the condition (1.5) for isochoric deformations we may express σ_2 and σ_3 in terms of σ_1. The ratios $\frac{\alpha_{13}}{\alpha_{12}}$, $\frac{\alpha_{21}}{\alpha_{23}}$, $\frac{\alpha_{32}}{\alpha_{31}}$ are expressed as non-constant functions of σ_1, which may be varied arbitrarily. A given ratio may thus be varied arbitrarily. It follows from (1.32) and (1.33) that

$$u_1 = u_2 = u_3 = 0 \ . \tag{8.6}$$

This argument fails if (8.4) holds.*

By (3.15) and (3.16), with (8.6), since none of the abnormalities can vanish,

$$x_1 = x_2 = x_3 = 0 \ , \tag{8.7}$$

and

$$y_1 = y_2 = y_3 = 0 \ . \tag{8.8}$$

By (8.6), (8.7), (8.8) and (3.18)

$$\operatorname{div} \underset{\sim}{c}_1 = \operatorname{div} \underset{\sim}{c}_2 = \operatorname{div} \underset{\sim}{c}_3 = 0 \ , \tag{8.9}$$

or by (1.35)

$$\pi_{32} = \pi_{23} \ , \quad \pi_{13} = \pi_{31} \ , \quad \pi_{21} = \pi_{12} \ . \tag{8.10}$$

By (3.11), (8.6), (8.7), (8.8) and (8.10)

$$2\, \pi_{11}\, \pi_{23} - 3\pi_{12}\, \pi_{31} = 0 \ , \quad etc. \tag{8.11}$$

and by (3.8), (3.9), (8.7), (8.8) and (8.10)

$$\frac{\delta}{\delta e_1} \pi_{31} = - \pi_{31}\, \pi_{23} + \pi_{12}\, \pi_{33} \ , \ etc. \tag{8.12}$$

---

*I am grateful to Professor Yin for his assistance with this argument.

and

$$\frac{\delta}{\delta e_1} \pi_{12} = \pi_{12} \pi_{23} - \pi_{31} \pi_{22} , \quad etc. \tag{8.13}$$

From (2.2) and (8.9)

$$\frac{\delta}{\delta e_1} \pi_{11} = 0 , \ etc. \tag{8.14}$$

Taking the gradient of (8.11) with respect to $\underset{\sim}{e}_1$, using (8.12), (8.13) and (8.14) we get

$$2\pi_{11} \frac{\delta}{\delta e_1} \pi_{23} = 3\pi_{12}^2 \pi_{33} - 3\pi_{31}^2 \pi_{22} , \quad etc. \tag{8.15}$$

From (4.8) or (4.9) with (8.6), (8.7), (8.8) and (8.9)

$$2\pi_{11} \frac{\delta}{\delta e_1} \pi_{23} + 2\pi_{22} \left[- \frac{\delta}{\delta e_2} \pi_{31} + \pi_{31}^2 \right]$$

$$- 2\pi_{33} \left[\frac{\delta}{\delta e_3} \pi_{12} + \pi_{12}^2 \right] = 0, \quad etc. \tag{8.16}$$

Substituting for $\frac{\delta}{\delta e_1} \pi_{23}$, *etc.*, from (8.15) into (8.16) we obtain

$$\pi_{12}^2 \pi_{33} - \pi_{31}^2 \pi_{22} = 0 , \quad etc. \tag{8.17}$$

The conditions (1.29), (1.30) take the form

$$\frac{\delta}{\delta e_1}\pi_{22} = -2\pi_{23}\pi_{22}\ , \quad etc.\ , \tag{8.18}$$

$$\frac{\delta}{\delta e_1}\pi_{33} = 2\pi_{32}\pi_{33}\ , \quad etc.\ , \tag{8.19}$$

and taking the gradient of (8.17) with respect to $\underset{\sim}{e}_1$ using (8.12), (8.13), (8.18) and (8.19), we obtain

$$\pi_{23}\pi_{31}^2\pi_{22} + \pi_{23}\pi_{12}^2\pi_{33} - \pi_{12}\pi_{31}\pi_{22}\pi_{33} = 0\ , \quad etc.\ , \tag{8.20}$$

By (8.17) and (8.20)

$$2\pi_{23}\pi_{12}^2\pi_{33} - \pi_{12}\pi_{31}\pi_{22}\pi_{33} = 0\ , \quad etc.\ , \tag{8.21}$$

Since π_{33} cannot vanish, it follows from (8.21) that either

$$\pi_{12} = 0\ , \quad etc.\ , \tag{8.22}$$

or else

$$2\pi_{12}\pi_{23} - \pi_{31}\pi_{22} = 0\ , \quad etc.\ , \tag{8.23}$$

and it follows from (8.11) and (8.23) that

$$\pi_{12} = \pi_{23} = \pi_{31} = 0\ . \tag{8.24}$$

Thus (8.24) holds without exception. It now follows from (8.14), (8.18) and (8.19) that all the abnormalities are constant. This is a contradiction. This proves Main Theorem I.2.

9. CONCLUDING REMARKS

The strongest special case we have been able to prove is that there are no new solutions when the ratios of the abnormalities are constant. This proof is given in the Appendix. To complete the proof that there are no new deformations we thus need to prove the existence of two independent polynomial relations among the three abnormalities and the proper numbers. Assuming that the ten conditions given by (5.2), (5.7), (6.9) and (6.10) are independent, then six further independent conditions would suffice to show that the ratios of the abnormalities are constant and the ratios $\frac{\sigma_1^2}{\sigma_2^2}$ and $\frac{\sigma_2^2}{\sigma_3^2}$ are constant.

Taking the gradient of (6.9) with respect to $\underset{\sim}{e}_3$, we obtain a higher order condition, cubic in u_a whose highest terms in u_a come from

$$u_3 \left[\frac{\alpha_{12}\alpha_{32}}{\alpha_{13}\alpha_{31}} \frac{(y_3 - 2x_3)u_3}{\pi_{13}} - \frac{(-x_3 + 2y_3)u_1}{\pi_{23}}\right] . \tag{9.1}$$

There are three such conditions obtained by cyclic permutation of the indices. A dual set related to (9.1) by the rotation condition is obtained by taking the gradient of (6.10) with respect to $\underset{\sim}{e}_3$, its highest terms in u_a come from

$$\frac{\alpha_{13}}{\alpha_{12}} u_1 \left[- u_1 \frac{(-x_3 + 2y_3)}{\pi_{23}} + \frac{\alpha_{12}\alpha_{32}}{\alpha_{13}\alpha_{31}} \frac{(y_3 - 2x_3)}{\pi_{13}} u_3\right]. \tag{9.2}$$

While the conditions whose leading terms are (9.1) and (9.2) appear to be independent of any conditions obtained by eliminating $\frac{\delta u_1}{\delta e_2}$, $\frac{\delta u_2}{\delta e_3}$, $\frac{\delta u_3}{\delta e_1}$ from (5.2), (5.7), (6.9) and (6.10) we cannot discount the possibility of lower order factors occurring at a later stage of the elimination. All one can say is that if these sixteen conditions are independent then there are no more solutions to the elasticity problem.

It seems likely that the development of the conditions (9.1) and (9.2) and the performances of the elimination necessary to establigh this independence could be accomplished using the computer symbolic mathematics systems [1979(1)], already referred to in Chapter 7. For the benefit of a reader who might wish to undertake this we tabulate the equations giving the various gradients required for deriving the conditions (9.1) and (9.2).

GRADIENT	EQUATION
$\frac{\delta}{\delta e_3}\pi_{21}$, $\frac{\delta}{\delta e_3}\pi_{12}$, *etc.*	(1.24), (1.26)
$\frac{\delta}{\delta e_3}\pi_{23}$, $\frac{\delta}{\delta e_3}\pi_{13}$, *etc.*	(1.32), (1.33)
$\frac{\delta}{\delta e_3}\pi_{32}$, $\frac{\delta}{\delta e_3}\pi_{31}$, *etc.*	(3.8), (3.9)
$\frac{\delta}{\delta e_3}\pi_{11}$, $\frac{\delta}{\delta e_3}\pi_{22}$, $\frac{\delta}{\delta e_3}\pi_{33}$, *etc.*	(1.29), (1.30), (2.2)
$\frac{\delta u_1}{\delta e_1}$, $\frac{\delta u_1}{\delta e_3}$, *etc.*	(3.15), (3.16)
$\frac{\delta x_1}{\delta e_1}$, $\frac{\delta x_1}{\delta e_2}$, $\frac{\delta x_1}{\delta e_3}$, *etc.*	(4.10), (4.1), (4.3)
$\frac{\delta y_1}{\delta e_1}$, $\frac{\delta y_1}{\delta e_2}$, $\frac{\delta y_1}{\delta e_3}$, *etc.*	(4.9), (4.2), (4.4)

APPENDIX

PROOF THAT NO NEW SOLUTIONS EXIST WHEN THE RATIOS OF THE ABNORMALITIES ARE CONSTANT

From (1972 [1]) and (1975 [1]) we assert that none of the abnormalities are zero and no two can be constant. We impose the conditions

$$\operatorname{grad}\left(\frac{\pi_{11}}{\pi_{22}}\right) = 0 \ , \qquad \operatorname{grad}\left(\frac{\pi_{22}}{\pi_{33}}\right) = 0 \ . \tag{A.1}$$

By (A.1), (1.29), (1.30), (1.35) and $(2.2)^1$, we obtain

$$u_1 = 2\pi_{13}\,\pi_{33} = -2\,\frac{\alpha_{21}}{\alpha_{23}}\,\pi_{31}\pi_{11} \ , \ etc. \ , \tag{A.2}$$

and hence, by (1.22),

$$\pi_{12}\,\pi_{23}\,\pi_{31} = \pi_{21}\,\pi_{32}\,\pi_{13} \ . \tag{A.3}$$

From (A.2) with (1.29) and (1.30) we also have

$$\frac{\delta}{\delta e_2}\,\pi_{33} = -\,2\pi_{33}\operatorname{div}\underset{\sim}{e}_2 \ , \quad \frac{\delta}{\delta e_2}\,\pi_{11} = -\,2\pi_{11}\operatorname{div}\underset{\sim}{e}_2 \ , \ etc.,$$

and incorporating these relations with $(2.2)^1$ we have the set of nine relations

$$\operatorname{div}\left(\log \pi_{aa}^{1/2}\,\underset{\sim}{e}_b\right) = 0 \ . \tag{A.4}$$

Again from (A.2) we have

$$\frac{\alpha_{12}}{\alpha_{13}} \pi_{13} u_3 + \pi_{23} u_1 = 0 \ , \quad etc. \tag{A.5}$$

which by (3.18) gives

$$\frac{\alpha_{23}}{\alpha_{21}} \pi_{33} x_1 - \frac{\alpha_{32}}{\alpha_{31}} \pi_{22} y_1 - \pi_{11}^2 \operatorname{div} \underset{\sim}{c}_1 = 0 \ , \ etc. \tag{A.6}$$

We prove

LEMMA 1. If one or more of the three curvatures π_{12}, π_{23}, π_{31} vanishes, there are no new solutions.

If all three curvatures vanish, then by (A.2) π_{21}, π_{32}, π_{13} all vanish. Thus div $\underset{\sim}{c}_1$, div $\underset{\sim}{c}_2$ are all zero. In this case by (A.4) all the abnormalities are constant, which is a contradiction.

If two curvatures vanish, say $\pi_{12} = \pi_{23} = 0$, then by (A.2) $\pi_{21} = \pi_{32} = 0$ and div $\underset{\sim}{c}_3$ and div $\underset{\sim}{c}_1$ are zero. From (1.32) and (1.33) we now get

$$\pi_{22} \pi_{13} = \frac{1}{2} \frac{\alpha_{13}}{\alpha_{12}} u_1$$

$$\pi_{22} \pi_{31} = \frac{1}{2} \frac{\alpha_{13}}{\alpha_{12}} u_1$$

from which it follows that $\pi_{13} = \pi_{31}$, so that div $\underset{\sim}{c}_2$ is zero. Once again the result follows from (A.4).

We conclude that for new solutions at most one of the three given curvatures vanishes. Suppose π_{31} vanishes. They by (A.3) π_{13} vanishes, so that div $\underset{\sim}{c}_2 = 0$. By (3.6), (3.8), (3.9) and (A.5) we have

$$\mathrm{div}\, \underset{\sim}{c}_2 = 0 \,, \qquad x_3 = y_1 = 0 \,, \;\; u_1 = 0 \,, \tag{A.7}$$

$$\left.\begin{aligned} x_1 &= - \pi_{32}\,\pi_{11} \,, \\ y_3 &= - \pi_{12}\,\pi_{33} \,. \end{aligned}\right] \tag{A.8}$$

By (A.7) and (1.33) we have

$$\frac{\delta}{\delta c_2}\,\pi_{32} = \pi_{33}\pi_{12} - \frac{1}{2}\,\frac{\alpha_{21}}{\alpha_{23}}\,u_2 \,, \tag{A.9}$$

and by (A.7) and (3.9) once again

$$\frac{\delta}{\delta c_2}\,\pi_{23} = - \pi_{33}\,\pi_{21} \,. \tag{A.10}$$

From (A.6), (A.7) and (A.8) with (1.35)

$$\frac{\alpha_{23}}{\alpha_{21}}\,\pi_{33}\pi_{32} + \pi_{11}\,(\pi_{23} - \pi_{32}) = 0$$

$$\frac{\alpha_{21}}{\alpha_{23}}\,\pi_{11}\pi_{12} - \pi_{33}\,(\pi_{12} - \pi_{21}) = 0 \quad . \tag{A.11}$$

From $(A.2)^2$, using (1.22) we have

$$\frac{\pi_{11}}{\pi_{33}} = - \frac{\alpha_{23}}{\alpha_{21}} \frac{\pi_{12}\pi_{23}}{\pi_{21}\pi_{32}} , \tag{A.12}$$

and substituting (A.12) into (A.11) we get the two conditions

$$\xi + \eta - \eta^2 = 0 ,$$

$$\xi + \eta - \xi^2 = 0 . \tag{A.13}$$

where $\xi = \frac{\pi_{21}}{\pi_{12}}$, $\eta = \frac{\pi_{23}}{\pi_{32}}$. Since ξ and η do not vanish, we must have

$$\frac{\pi_{21}}{\pi_{12}} = \frac{\pi_{23}}{\pi_{32}} = 2 . \tag{A.14}$$

From (A.10) and (A.14)

$$\frac{\delta}{\delta e_2} \pi_{32} = - \pi_{33}\pi_{12} , \tag{A.15}$$

and by (A.9) and (A.15)

$$u_2 = .4 \frac{\alpha_{23}}{\alpha_{21}} \pi_{12}\pi_{33} ,$$

but by (A.2)

$$u_2 = - 2 \frac{\alpha_{32}}{\alpha_{31}} \pi_{12}\pi_{22} ,$$

so that by (1.22)

$$\pi_{33} - \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}\pi_{22} = 0 \ . \tag{A.16}$$

But by $(A.2)^2$

$$\frac{\pi_{32}}{\pi_{23}} = -\frac{\alpha_{13}}{\alpha_{12}}\frac{\pi_{33}}{\pi_{22}}$$

so that by (A.14)

$$\pi_{33} + \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}\pi_{22} = 0 \ . \tag{A.17}$$

By (A.16) and (A.17) $\pi_{22} = \pi_{33} = 0$, which is a contradiction. This completes the proof of Lemma 1.

From Lemma 1 and (A.3) we immediately have

LEMMA 2. *If one or more of the three curvatures π_{21} π_{32} π_{13} vanishes there are no new solutions.*

We prove

LEMMA 3. *If there are new solutions then the ratios $\frac{\pi_{12}}{\pi_{11}}$, $\frac{\pi_{23}}{\pi_{11}}$, $\frac{\pi_{31}}{\pi_{11}}$ must all be constant.*

From (A.2) the ratio $\frac{\pi_{13}}{\pi_{31}}$ is constant, so that

$$\pi_{31}\frac{\delta}{\delta e_2}\pi_{13} - \pi_{13}\frac{\delta}{\delta e_2}\pi_{31} = 0 \ ,$$

or, by (A.2)

$$\frac{\pi_{33}}{\pi_{11}}\frac{\delta}{\delta e_2}\pi_{13} + \frac{\pi_{21}}{\pi_{23}}\frac{\delta}{\delta e_2}\pi_{31} = 0 \ . \tag{A.18}$$

Substituting from (1.24), (1.25), (1.26) and (1.27) into (A.18), using $\pi_{13} = -\frac{\alpha_{21}}{\alpha_{23}} \frac{\pi_{11}}{\pi_{33}} \pi_{31}$, *etc.*, as given by (A.2) and using (1.22), we obtain from (A.18)

$$\frac{\alpha_{31}\alpha_{13}}{\alpha_{32}\alpha_{23}} \frac{\pi_{33}}{\pi_{22}} \theta_1^2 + \frac{\alpha_{21}}{\alpha_{23}} \left[\frac{\alpha_{21}}{\alpha_{23}} \frac{\pi_{11}}{\pi_{33}} - 1\right] \theta_2^2 + \frac{\alpha_{32}}{\alpha_{31}} \frac{\pi_{22}\pi_{33}}{\pi_{11}^2} \theta_3^2$$

$$+ \frac{\pi_{33}}{\pi_{11}^3} \xi_{213} + \frac{\alpha_{21}}{\alpha_{23}} \frac{\eta_{231}}{\pi_{11}^2} = 0 \tag{A.19}$$

where

$$\theta_1 = \frac{\pi_{23}}{\pi_{11}}, \quad \theta_2 = \frac{\pi_{31}}{\pi_{11}}, \quad \theta_3 = \frac{\pi_{12}}{\pi_{11}} .$$

Since none of the curvatures vanish we have three equations (A.19) obtained by cyclic permutation of the indices. Calculating the discriminant of these equations we find that the determinant has the value 4. The terms $\frac{\pi_{33}}{\pi_{11}^3} \xi_{213} + \frac{\alpha_{21}}{\alpha_{23}} \frac{\eta_{231}}{\pi_{11}^2}$ involve ratios of the abnormalities and are constant. It follows that θ_1, θ_2, θ_3 are constant. The constancy of the ratios $\frac{\pi_{32}}{\pi_{11}}$, $\frac{\pi_{13}}{\pi_{11}}$, $\frac{\pi_{21}}{\pi_{11}}$ then follows from (A.2). This proves Lemma 3.

We prove

LEMMA 4. At least one of the quantities (2.7) must be zero.

Since $\frac{\pi_{21}}{\pi_{11}}$ is constant, $\frac{\delta}{\delta e_1} \frac{\pi_{21}}{\pi_{11}} = 0$, and it follows from (1.33), (1.35), $(2.2)^1$ that

$$\pi_{22}\pi_{31} + \pi_{23}(\pi_{21} - \pi_{12}) - 2\pi_{21}\pi_{32} - \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}} u_1 = 0 .$$

Using $(A.2)^{1,2}$ to eliminate u_1 and to eliminate π_{21} in favor of π_{12} we now obtain using (1.22),

$$\left(\pi_{22} - \frac{\alpha_{31}}{\alpha_{32}}\pi_{11}\right)\pi_{31} - \left(1 + \frac{\alpha_{32}}{\alpha_{31}}\frac{\pi_{22}}{\pi_{11}} - 2\frac{\alpha_{23}}{\alpha_{21}}\frac{\pi_{33}}{\pi_{11}}\right)\pi_{23}\pi_{12} = 0 . \qquad (A.20)$$

Since $\frac{\pi_{23}}{\pi_{33}}$ is constant, $\frac{\delta}{\delta e_3}\frac{\pi_{23}}{\pi_{33}} = 0$. Applying the same argument as before only now using the expression (1.32) for $\frac{\delta}{\delta e_3}\pi_{23}$ we get

$$\frac{\pi_{11}}{\pi_{33}}\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{22} + \frac{\alpha_{31}}{\alpha_{32}}\pi_{33}\right]\pi_{31}$$

$$+ \left[2 + \frac{\alpha_{32}}{\alpha_{31}}\left(\frac{\alpha_{13}}{\alpha_{12}}\frac{\pi_{33}}{\pi_{11}} + \frac{\pi_{22}}{\pi_{11}}\right)\right]\pi_{23}\pi_{12} = 0 . \qquad (A.21)$$

We require that (A.20) and (A.21) be solvable for non-zero values of π_{31} and $\pi_{23}\pi_{12}$ (Lemma 1). Using (1.22) we find that the determinant reduces to

$$\left(\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right)\left(\pi_{11} + \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}\right)\left(\pi_{22} + \frac{\alpha_{31}}{\alpha_{32}}\pi_{11}\right) = 0 . \qquad (A.22)$$

This proves Lemma 4.

According to Lemma 4 we take

$$\lambda_1 = \pi_{33} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22} = 0 \ , \tag{A.23}$$

and it follows from $(A.2)^2$ that

$$\operatorname{div} \underset{\sim}{e}_1 = 0 \ , \qquad \pi_{23} = \pi_{32} \ . \tag{A.24}$$

Also by $(A.2)^2$

$$\pi_{31}\lambda_2 = \frac{\alpha_{23}}{\alpha_{21}}\pi_{33} \operatorname{div} \underset{\sim}{e}_2 \ , \tag{A.25}$$

$$\pi_{12}\lambda_3 = \frac{\alpha_{31}}{\alpha_{32}}\pi_{11} \operatorname{div} \underset{\sim}{e}_3 \ . \tag{A.26}$$

We have

Lemma 5. If either div $\underset{\sim}{e}_2$ or div $\underset{\sim}{e}_3$ vanishes there are no new solutions.

Suppose div $\underset{\sim}{e}_2$ vanishes, then by (A.25) and Lemma 1, $\lambda_2 = 0$. It then follows by (A.23) and (1.22) that $\lambda_3 = 0$, so that by (A.26) div $\underset{\sim}{e}_3 = 0$. By (A.4) the abnormalities are constant. This is a contradiction.

Equivalently we have

Lemma 6. If either λ_2 or λ_3 vanishes there are no new solutions.

By (A.23), (A.25), (A.26) and $(A.2)^1$, the condition (2.6) is reduced to

$$- \operatorname{div} \underset{\sim}{e}_2\, \underset{\sim}{e}_2 - \operatorname{div} \underset{\sim}{e}_3\, \underset{\sim}{e}_3 = \operatorname{grad} \log \pi_{11}^{1/2} \tag{A.27}$$

a condition which is in accord with (A.4). Since all the curvatures and abnormalities are constant on the surface π_{11} = constant, and since by (A.27) the vector-lines of $\underset{\sim}{e}_1$ lie on these surfaces, we have

Lemma 7. The vector-lines of $\underset{\sim}{e}_1$ are circular helices and the surfaces π_{11} = constant are concentric circular cylinders.

From (A.24) and (3.6) one sees that

$$x_2 = 0\ , \quad y_3 = 0\ , \tag{A.28}$$

and by (3.8) and (3.9)

$$\begin{aligned} -\pi_{23}\pi_{13} + \pi_{33}\pi_{12} &= 0\ , \\ -\pi_{22}\pi_{13} + \pi_{32}\pi_{12} &= 0\ , \end{aligned} \tag{A.29}$$

and for (A.29) to give non-zero values of π_{12} and π_{13} in accordance with Lemma 1 and (A.2) we must have

$$\pi_{22}\pi_{33} = \pi_{23}\pi_{32} = \pi_{23}^2 \quad , \quad \text{by (A.24)} . \tag{A.30}$$

From (A.3) and (A.24)

$$\pi_{13}\pi_{21} - \pi_{31}\pi_{12} = 0 .$$

These conditions are combined in the form

$$\frac{\pi_{13}}{\pi_{12}} = \frac{\pi_{33}}{\pi_{23}} = \frac{\pi_{23}}{\pi_{22}} = \frac{\pi_{31}}{\pi_{21}} . \tag{A.31}$$

From (A.6), (A.23) and (A.28)

$$x_3 = -\pi_{33}\,\text{div}\,\underset{\sim}{e}_3 , \quad y_2 = \pi_{22}\,\text{div}\,\underset{\sim}{e}_2 . \tag{A.32}$$

The condition (A.6) with (1.22), (A.23) and (A.24) gives $x_1 = y_1$. Using $\frac{\delta}{\delta e_2}\left(\frac{\pi_{21}}{\pi_{22}}\right) = 0$ with (1.35), (2.2)[1] and (3.8) we then get

$$x_1 = y_1 = \pi_{21}(2\pi_{13} - \pi_{31}) - \pi_{11}\pi_{23} \tag{A.33}$$

and it follows from (3.6), (A.31) and (A.33) that

$$\frac{\delta}{\delta e_2}\,\text{div}\,\underset{\sim}{e}_3 = \frac{\delta}{\delta e_3}\,\text{div}\,\underset{\sim}{e}_2 = \pi_{13}\pi_{21} - \pi_{11}\pi_{23} . \tag{A.34}$$

We note that the conditions (1.24) to (1.27) may be written

$$\frac{\delta}{\delta e_3}\pi_{21} = (\pi_{21}^2 - \pi_{11}\pi_{22}) - \frac{\alpha_{12}}{\alpha_{32}}(\pi_{32}\pi_{23} - \pi_{22}\pi_{33})$$

$$- \frac{\alpha_{23}}{\alpha_{21}}\lambda_3\pi_{33} + \frac{3}{4}\lambda_2\lambda_3 + \frac{1}{4}\left[\frac{\alpha_{23}}{\alpha_{21}} - \frac{\alpha_{12}}{\alpha_{32}}\right]\lambda_3\lambda_1$$

$$- \frac{1}{4}\frac{\alpha_{13}\alpha_{21}}{\alpha_{32}\alpha_{23}}\lambda_1\lambda_2 \;, \quad etc. \;, \tag{A.35}$$

and

$$\frac{\delta}{\delta e_2}\pi_{31} = (\pi_{11}\pi_{33} - \pi_{31}^2) - \frac{\alpha_{13}}{\alpha_{23}}(\pi_{22}\pi_{33} - \pi_{23}\pi_{32})$$

$$- \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\lambda_2 + \frac{3}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3 + \frac{1}{4}\frac{\alpha_{13}}{\alpha_{23}}\lambda_3\lambda_1$$

$$+ \frac{1}{4}\left[1 - \frac{\alpha_{13}\alpha_{31}}{\alpha_{32}\alpha_{23}}\right]\lambda_1\lambda_2 \;, \quad etc. \tag{A.36}$$

Since all the curvatures bear constant values on the vector-lines of $\underline{e}_1$, we have in particular $\frac{\delta}{\delta e_1}\pi_{32} = 0$, $\frac{\delta}{\delta e_1}\pi_{23} = 0$. Using (A.23). (A.30), and (1.22) we obtain from (A.35) and (A.36)

$$\pi_{33}\pi_{11} - \pi_{13}\pi_{31} = -\frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3 \;, \tag{A.37}$$

$$\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = \frac{1}{4}\lambda_2\lambda_3 \;. \tag{A.38}$$

Again by (A.35) and (A.36)

$$\frac{\delta}{\delta e_3} \operatorname{div} \underset{\sim}{e}_3 = \frac{\delta}{\delta e_3}(\pi_{12} - \pi_{21}) = (\pi_{22}\pi_{11} - \pi_{12}^2) - \frac{\alpha_{21}}{\alpha_{31}}(\pi_{33}\pi_{11} - \pi_{31}\pi_{13})$$

$$- \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}\lambda_3 + \frac{1}{4}\left[1 - \frac{\alpha_{21}\alpha_{12}}{\alpha_{13}\alpha_{31}}\right]\lambda_2\lambda_3 - (\pi_{21}^2 - \pi_{11}\pi_{22})$$

$$+ \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}\lambda_3 - \frac{3}{4}\lambda_2\lambda_3 ,$$

and by (A.37) and (A.38) we obtain

$$\frac{\delta}{\delta e_3} \operatorname{div} \underset{\sim}{e}_3 = -(\operatorname{div} \underset{\sim}{e}_3)^2 . \qquad \text{(A.39)}$$

Similarly we obtain

$$\frac{\delta}{\delta e_2} \operatorname{div} \underset{\sim}{e}_2 = -(\operatorname{div} \underset{\sim}{e}_2)^2 . \qquad \text{(A.40)}$$

From (A.27), (A.39) and (A.40) we note

Lemma 8. The function $\log \pi_{11}^{1/2}$ is a cylindrical harmonic function.

Let $\underset{\sim}{e}_3^*$ be the unit normal to the surface $\log \pi_{11}^{1/2}$ = constant, then $\underset{\sim}{e}_2^* = \underset{\sim}{e}_3^* \times \underset{\sim}{e}_1$ is the unit vector perpendicular to $\underset{\sim}{e}_1$ in the tangent plane of the surface. Then

$$\frac{\delta}{\delta e_2^*} \operatorname{div} \underset{\sim}{e}_2 = 0 , \qquad \frac{\delta}{\delta e_2^*} \operatorname{div} \underset{\sim}{e}_3 = 0 ,$$

and

$$\underset{\sim}{e}_2^* = \frac{-\operatorname{div} \underset{\sim}{e}_3 \, \underset{\sim}{e}_2 + \operatorname{div} \underset{\sim}{e}_2 \, \underset{\sim}{e}_3}{D} \quad .$$

We obtain from (A.39) and (A.40)

$$\operatorname{div} \underset{\sim}{e}_2 \left[\frac{\delta}{\delta e_3} \operatorname{div} e_2 + \operatorname{div} e_2 \operatorname{div} e_3\right] = 0 \ ,$$

$$\operatorname{div} \underset{\sim}{e}_3 \left[\frac{\delta}{\delta e_2} \operatorname{div} e_3 + \operatorname{div} e_2 \operatorname{div} e_3\right] = 0 \ ,$$

and by Lemma 5

$$\frac{\delta}{\delta e_3} \operatorname{div} \underset{\sim}{e}_2 = \frac{\delta}{\delta e_2} \operatorname{div} \underset{\sim}{e}_3 = - \operatorname{div} \underset{\sim}{e}_2 \operatorname{div} \underset{\sim}{e}_3 \ . \tag{A.41}$$

From (A.34) and (A.41)

$$- \operatorname{div} \underset{\sim}{e}_2 \operatorname{div} \underset{\sim}{e}_3 = \pi_{13}\pi_{21} - \pi_{11}\pi_{23} \ . \tag{A.42}$$

By (A.25), (A.26), and (A.42), using (1.22) we obtain

$$\frac{\alpha_{13}}{\alpha_{12}} \pi_{11}\pi_{33} \left(\pi_{13}\pi_{21} - \pi_{11}\pi_{23}\right) - \pi_{12}\pi_{31} \lambda_2\lambda_3 = 0 \ . \tag{A.43}$$

We write

$$\pi_{13}\pi_{21} = \pi_{12}\pi_{31} \stackrel{\text{def}}{=} x \ , \qquad \pi_{23} \stackrel{\text{def}}{=} y \ , \tag{A.44}$$

and (A.43) becomes

$$\left[\pi_{11}\pi_{33} - \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3\right] x - \pi_{11}^2\pi_{33} y = 0 . \tag{A.45}$$

From (A.31) $\pi_{13}\pi_{31} = \pi_{33} \left(\frac{\pi_{21}\pi_{13}}{\pi_{23}}\right)$ so that

$$\pi_{13}\pi_{31} = \pi_{33} \frac{x}{y} ,$$

and we obtain from (A.37)

$$- \pi_{33}x + \left[\pi_{11}\pi_{33} + \frac{1}{4} \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3\right] y = 0 . \tag{A.46}$$

For (A.45) and (A.46) to give non-vanishing values for x and y one must have

$$\pi_{11}^2\pi_{33}^2 - \left[\pi_{11}\pi_{33} - \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3\right] \left[\pi_{11}\pi_{33} + \frac{1}{4} \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3\right] = 0$$

so that either

$$\lambda_2\lambda_3 = 0 , \tag{A.47}$$

or

$$3\pi_{11}\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3 = 0 . \tag{A.48}$$

If (A.47) holds the result follows from Lemma 6.

We are left with the possibility (A.48). Using the expressions (2.7) for λ_2 and λ_3, and eliminating π_{33} in favor of π_{22} by (A.23), and using (1.22) we reduce (A.48) to

$$- \pi_{11}\pi_{22} + \frac{\alpha_{31}}{\alpha_{32}} \pi_{11}^2 + \frac{\alpha_{32}}{\alpha_{31}} \pi_{22}^2 = 0 . \tag{A.49}$$

We may now obtain an expression for $\frac{\delta}{\delta e_2} \pi_{13}$ which is independent of that given by (A.35) by taking the gradient of (A.42) with respect to $\underset{\sim}{e}_2$. From (A.40) and (A.41) we have

$$\frac{\delta}{\delta e_2} (\text{div } e_2 \text{ div } e_3) = - 2(\text{div } \underset{\sim}{e}_2)^2 \text{ div } \underset{\sim}{e}_3 , \tag{A.50}$$

and by (1.35), (3.9) and (A.32)[1]

$$\frac{\delta}{\delta e_2} \pi_{23} = - x_3 + \pi_{23}\pi_{13} - \pi_{21}\pi_{33} = 2\pi_{33} \text{ div } \underset{\sim}{e}_3 . \tag{A.51}$$

Also by (A.33) and (3.8)

$$\frac{\delta}{\delta e_2} \pi_{21} = - \pi_{21} \text{ div } \underset{\sim}{e}_2 . \tag{A.52}$$

By (A.4), (A.50), (A.51) and (A.52) we obtain for the $\underset{\sim}{e}_2$-gradient of (A.42)

$$2 \ (\text{div } e_2)^2 \text{ div } e_3 = \left(\frac{\delta}{\delta e_2} \pi_{13}\right) \pi_{21} - 2(\pi_{13}\pi_{21} - \pi_{11}\pi_{23}) \text{ div } \underset{\sim}{e}_2$$

$$- 2\pi_{11}\pi_{33} \text{ div } \underset{\sim}{e}_3 ,$$

which by (A.42) reduces to

$$\pi_{21} \frac{\delta}{\delta e_2} (\pi_{13}) - 2\pi_{11}\pi_{33} \operatorname{div} \underset{\sim}{e}_3 = 0 . \tag{A.53}$$

By $(A.2)^2$ and (A.52), eliminating π_{12} in favor of π_{21} and then cancelling π_{21} which by Lemma 2 does not vanish, we get

$$\pi_{22} \left(\frac{\delta}{\delta e_2} \pi_{13} + 2 \pi_{11}\pi_{33} \right) + 2 \frac{\alpha_{31}}{\alpha_{32}} \pi_{11}^2 \pi_{33} = 0 , \tag{A.54}$$

which is the required expression for $\frac{\delta}{\delta e_2} \pi_{13}$.

By (A.35) and (A.38) for $\lambda_1 = 0$

$$\frac{\delta}{\delta e_2} \pi_{13} = \pi_{13}^2 - \pi_{33}\pi_{11} - \frac{\alpha_{12}}{\alpha_{13}} \pi_{22}\lambda_2 + \frac{1}{4} \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3 . \tag{A.55}$$

Also by (A.2)

$$\pi_{13}^2 = - \frac{\alpha_{21}}{\alpha_{23}} \frac{\pi_{11}}{\pi_{33}} \pi_{13}\pi_{31}$$

$$= - \frac{\alpha_{21}}{\alpha_{23}} \frac{\pi_{11}}{\pi_{33}} \left[\pi_{33}\pi_{11} + \frac{1}{4} \frac{\alpha_{12}}{\alpha_{13}} \lambda_2\lambda_3 \right] , \text{ by (A.37)} , \tag{A.56}$$

Eliminating π_{13}^2 from (A.55) by means of (A.56) and then substituting for $\frac{\delta}{\delta e_2} \pi_{13}$ in (A.53) we get

$$\pi_{22}\left[-\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2+\frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\left(1-\frac{\pi_{11}}{\pi_{33}}\frac{\alpha_{21}}{\alpha_{23}}\right)\lambda_2\lambda_3+\pi_{33}\pi_{11}-\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\lambda_2\right]$$

$$+\frac{2\alpha_{31}}{\alpha_{32}}\pi_{11}^2\pi_{33}=0\ . \tag{A.57}$$

By (A.23) and (1.22)

$$2\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}^2\pi_{33}=-2\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}^2\pi_{22}=2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2\pi_{22}\ ,$$

so (A.57) reduces to

$$\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2+\frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\left(1-\frac{\pi_{11}}{\pi_{33}}\frac{\alpha_{21}}{\alpha_{23}}\right)\lambda_2\lambda_3+\pi_{33}\pi_{11}$$

$$-\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\pi_{22}=0\ . \tag{A.58}$$

Substituting for $\lambda_2\lambda_3$ from (A.48), substituting for λ_2, and eliminating π_{33} in favor of π_{22} by (A.23) we reduce (A.58) to

$$\frac{7}{4}\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2-\frac{5}{4}\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}\pi_{22}-\frac{\alpha_{32}\alpha_{12}}{\alpha_{31}\alpha_{13}}\pi_{22}^2=0\ . \tag{A.59}$$

Eliminating π_{22}^2 between (A.49) and (A.59) we obtain, using (1.22)

$$\pi_{11}=-3\frac{\alpha_{32}}{\alpha_{31}}\pi_{22}\ . \tag{A.60}$$

Substituting (A.60) into (A.49) we get

$$13 \frac{\alpha_{32}}{\alpha_{31}} \pi_{22}^{2} = 0 . \tag{A.61}$$

This proves the result.

ACKNOWLEDGMENTS

The author expresses his appreciation to the U. S. National Science Foundation who supported this study by grant. He is indebted to Mrs. Van Hook for her care and patience in typing this work.

The writer wishes particularly to acknowledge the help of Professor Wan-Lee Yin. Through his long experience with this problem he has repeatedly saved the writer from emotionally inspired errors in logic.

REFERENCES

1954 [1] ERICKSEN, J. L., "Deformations possible in every isotropic incompressible perfectly elastic body", *Z. angew. Math. Phys.* 5, 466-486.

1965 [1] SINGH, M. AND PIPKIN, A. C., "Note on Ericksen's problem", *Z. angew. Math. Phys.* 16, 706-709.

1966 [1] FOSDICK, R. L., "Remarks on compatibility", *Modern Developments in the Mechanics of Continua*. Academic Press, New York, 109-127.

1966 [2] KLINGBEIL, W. W. and SHIELD, R. T., "On a class of solutions in plane finite elasticity", *Z. angew. Math. Phys.* 17, 489-511.

1969 [1] FOSDICK, R. L. and SCHULER, K. W., "On Ericksen's problem for plane deformations with uniform transverse stretch", *Int. J. Eng. Science* 7, 217-233.

1970 [1] MARRIS, A. W. and SHIAU, J. F., "Universal deformations in isotropic incompressible hyperelastic materials when the deformation tensor has equal proper values", *Arch. Rational Mech. Anal.* 36, 135-160.

1971 [1] FOSDICK, R. L., "Statically possible radially symmetric deformations in isotropic, incompressible elastic solids", *Journal of Applied Mathematics and Physics (ZAMP)* 22, 590-607.

1972 [1] KAFADAR, C. B., "On Ericksen's problem", *Arch. Rational Mech. Anal.*, 47, 15-27.

1973 [1] MARRIS, A. W., "Hamel's Theorem", *Arch. Rational Mech. Anal.* 51, 85-105.

1975 [1] MARRIS, A. W., "Universal deformations in incompressible isotropic elastic materials", *Journal of Elasticity*, *5*, 111-128.

1979 [1] STOUTEMYER, D. R., "Symbolic computation comes of age", *SIAM NEWS*, *12*, 6, 1-2.

NATIONAL SCIENCE FOUNDATION Washington, D.C. 20550	FINAL PROJECT REPORT NSF FORM 98A

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Tech Research Institute 225 North Avenue NW Atlanta, Georgia 30332	Fluid Mechanics	CME-7820240
	4. Award Period From 4-1-80 To 9-30-82	5. Cumulative Award Amount $66,610

6. Project Title

"STUDIES ON CONTROLLABLE MOTIONS"

PART II—SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

The writer published the paper "Remarks on Plane Universal Navier-Stokes Motions" (Arch. Rational Mech. Anal. Vol 77 p 95-102). Motivated by this work, Professor Stallybrass of the Mathematics department of Georgia Institute of Technology discovered some new solutions of Navier-Stokes equations representing unsteady plane universal motions. Professor Stallybrass's paper is currently in press for Letters in Applied and Engineering Science. Subsequently the writer's colleague, Professor Wan-Lee Yin, has determined completely the set of unsteady plane universal motions of incompressible Navier-Stokes fluids. Professor Yin's research, settling this problem, will be submitted for publication in the near future.

Invoking the theorem that lamellar-solenoidal vector field whose vector magnitude is constant on a vector line must be helical fields, (a result originally given with an unsatisfactory proof by Hamel, but subsequently proved by the author), the writer has delimited completely the class of isochoric circulation-preserving motions whose vorticity is steady and bears a constant magnitude on a vortex-line. This work is published in ZAMP Vol 33 pp 124-131.

Ericksen's problem concerns the universal deformations of classical finite elasticity in the case when the proper numbers of the deformation tensor are constant. The problem is to determine all the new solutions or to prove that no further solutions exist. In the long paper "Two New Theorems on Ericksen's Problem", Arch. Rational Mech. Anal. Vol 79, No. 2, p 131-175) the writer has proved that if a new solution exist it must be determined by a polynomial relation among the curvatures and abnomalities of the proper vector field and the ratios of the proper numbers. It is also proved that no new solutions are possible if the ratios of the abnormalities are constant. Implicit in this work is the demonstration that the equilibrium conditions are not wholly independent of the geometric (compatibility) conditions, this emphasizes the ultimate weakness of the defining conditions and the difficulty of the problem. A formalism is established which may form a basis for a computer symbolic program to determine the

PART III—TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (✓)	Approx. Date
a. Abstracts of Theses	✓				
b. Publication Citations	✓				
c. Data on Scientific Collaborators					
d. Information on Inventions					
e. Technical Description of Project and Results		✓			
f. Other *(specify)* Reprints and manuscript of publications		✓			

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
Andrew W. Marris		9-8-82

Form Approved OMB No. 99R0013

remaining high order relations required to disclose new solutions or establish that none exist.

Screw motions are motions in which the vorticity and the velocity are parallel vectors. Earlier the writer showed that there are no solutions to the Navier-Stokes equations for incompressible fluids determining steady screw motions (Arch. Rational Mech. Anal. Vol 70 p 47-60). There is however a substantical class of screw motions which, while unsteady, do posses steady stream-line, which are Navier-Stokes flows. There are Trkalian flows, characterized by the condition that the stream-line abnormality is constant. The writer has now proved that Trkalian motions are the only unsteady screw motions with steady stream-lines possible for classical incompressible viscous fluids. This work entitled "On the Completeness of Trkalian Motions" is currently in press for the Archive for Rational Mechanics and Analysis.

ON THE COMPLETENESS OF TRKALIAN MOTIONS

OF NAVIER-STOKES FLUIDS

A. W. Marris

School of Engineering Science and Mechanics
Georgia Institute of Technology
Atlanta, Georgia 30332

March 1982

TABLE OF CONTENTS

INTRODUCTION

Consider a motion whose velocity field is $\underset{\sim}{v} = v\underset{\sim}{e}_1$, where v is the velocity magnitude and $\underset{\sim}{e}_1$ is the unit vector tangent to the stream line. The motion is called a screw motion* if the velocity $\underset{\sim}{v}$ and the vorticity $\underset{\sim}{\omega} = \text{curl } \underset{\sim}{v}$ are parallel vectors, so that

$$\underset{\sim}{\omega} \times \underset{\sim}{v} = 0 , \tag{I.1}$$

or equivalently

$$\underset{\sim}{\omega} = \pi_{11}\underset{\sim}{v} = \pi_{11}v\underset{\sim}{e}_1 , \tag{I.2}$$

where $\pi_{11} = \underset{\sim}{e}_1 \cdot \text{curl } \underset{\sim}{e}_1$ is the abnormality of the vector-lines of $\underset{\sim}{v}$. The vector-lines of $\underset{\sim}{v}$ and $\underset{\sim}{\omega}$ are coincident.

A screw motion will be possible for a Navier-Stokes fluid,** if in addition to (I.1) the following conditions hold

$$\text{div } \underset{\sim}{v} = 0 , \tag{I.3}$$

and

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} = \quad \nu \text{ curl curl } \underset{\sim}{\omega} , \tag{I.4}$$

where ν is the kinematic viscosity.

In an earlier paper (1979), this writer has shown that no steady rotational screw motions of Navier-Stokes fluids exist. However, there is a class of screw motions of Navier-Stokes fluids, which are unsteady motions with steady stream lines.

*These motions are also known as Beltrami motions, see Bjørgum (1951) and Bjørgum and Godal (1952).

**i.e., an incompressible viscous fluid with uniform density and viscosi

Consider the class of screw motions in which the abnormality π_{11} is spatially constant.* These motions, known as Trkalian motions, have been the subject of a treatise by Bjørgum and Godal (1952). It is shown that for a Trkalian motion to be a possible motion of a Navier Stokes fluid the abnormality π_{11} must also be temporarily constant and that the velocity field is given by

$$\underset{\sim}{v} = e^{-\nu\pi_{11}^{2}t}\,\underset{\sim}{u} \tag{I.5}$$

where $\underset{\sim}{u}$ is any steady Trkalian field of abnormality π_{11} (see (1952) p. 9-11). Evidently by (1.2) curl $\underset{\sim}{u} = \pi_{11}\underset{\sim}{u}$. Equation (I.5) defines a motion whose stream lines are steady but whose velocity magnitude decays exponentially with time.

Motions with steady stream lines have some of the physical significance of steady motions. It seems natural to ask, are Trkalian motions the only rotational screw motions of Navier-Stokes fluids for which the stream lines are steady?

In this paper we show that the answer is yes. We prove

THEOREM. The only rotational screw motions of Navier-Stokes fluids for which the stream lines are steady are Trkalian motions characterized by the spatially constant abnormality π_{11}.

*An equivalent condition is that the vorticity $\underset{\sim}{\omega}$ must also be a screw field.

It follows from (I.2)[1] that

$$\underset{\sim}{v} \times \frac{\partial \underset{\sim}{v}}{\partial t} = \frac{1}{\Omega^2}\, \underset{\sim}{\omega} \times \frac{\partial \underset{\sim}{\omega}}{\partial t} \ . \tag{I.6}$$

A necessary and sufficient condition for a motion with steady stream lines is $\underset{\sim}{v} \times \frac{\partial \underset{\sim}{v}}{\partial t} = 0$. It then follows from (I.4) and (I.6) that for screw motions of Navier Stokes fluids with steady stream lines it is necessary that

$$\underset{\sim}{\omega} \times \text{curl curl } \underset{\sim}{\omega} = 0 \ , \tag{I.7}$$

or, by (I.2)

$$\underset{\sim}{e}_1 \times \text{curl curl } \underset{\sim}{\omega} = 0 \ . \tag{I.8}$$

Our analysis is based primarily on the governing equation (I.8)*. As is to be expected it is considerably more involved than the author's analysis of steady screw motions of Navier Stokes fluids (1979), an analysis which was based on the full condition curl curl $\underset{\sim}{\omega} = \underset{\sim}{0}$.

Early on in the investigation it appeared that there might be a special solution in which the steady stream lines are circular helices. This flow is shown to be impossible in Chapter 3.

---

*The condition

$$\underset{\sim}{e}_1 \cdot \frac{\partial \underset{\sim}{\omega}}{\partial t} = -\nu \ \underset{\sim}{e}_1 \cdot \text{curl curl } \underset{\sim}{\omega}$$

representing the flow-wise resolute of the system (I.4), is not generally helpful. See Chapter 3 following Equation (3.30) and the section 'Concluding Remarks' at the end of the paper.

The main proof is based on two Lemmas. For the variables $\theta, \psi, \kappa, \tau$ defined in Chapter 2, we show that for $\kappa \neq 0$, $\tau \neq 0$ and λ not constant, there exist a functional relation $f\left(\frac{\kappa}{\tau}, \frac{\theta-\psi}{\tau}\right) = 0$. This is Lemma 1. We then invoke the Jacobian theorem for functional dependence to obtain two further somewhat simpler necessary conditions. We are then able to establish Lemma 2, asserting that there exist a functional relation $f(\theta-\psi, \lambda, \tau) = 0$ where $\lambda = -3\frac{\delta\tau}{\delta s}$, (equation (2.36)). The result follows directly from Lemma 2.

1. PRELIMINARY SPECIFICATION OF THE VECTOR FIELD

Since the case of spatially constant π_{11} determines Trkalian motions, we postulate that π_{11} is not spatially constant. Since the vorticity $\underset{\sim}{\omega}$ is solenoidal it follows from (I.2) and (I.3) that*

$$\underset{\sim}{e}_1 \cdot \text{grad}\, \pi_{11} = \frac{\delta \pi_{11}}{\delta e_1} = 0 \ . \tag{1.1}$$

The family of surfaces π_{11} = constant contain the vector lines of $\underset{\sim}{e}_1$. We may define the unit vector $\underset{\sim}{e}_2$ to be normal to these surfaces so that

$$\underset{\sim}{e}_2 = \lambda \text{ grad } \pi_{11} \ . \tag{1.2}$$

Hence

$$\pi_{22} = \underset{\sim}{e}_2 \cdot \text{curl } \underset{\sim}{e}_2 = 0 \ . \tag{1.3}$$

The vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$, where $\underset{\sim}{e}_3$ is perpendicular to $\underset{\sim}{e}_1$ in the tangent plane to the surface π_{11} = constant, form an ortho-normal basis, whence one has

$$\frac{\delta \pi_{11}}{\delta e_3} = 0 \ . \tag{1.4}$$

---

*I use the notation $\frac{\delta}{\delta e_1}$, $\frac{\delta}{\delta e_2}$, $\frac{\delta}{\delta e_3}$ to denote the components of gradients, thus $\frac{\delta F}{\delta e_1} = \underset{\sim}{e}_1 \cdot \text{grad } F$ and $\frac{\delta^2 F}{\delta e_1 \delta e_2} = \underset{\sim}{e}_1 \cdot \text{grad}(\underset{\sim}{e}_2 \cdot \text{grad } F)$, and so on. The symbols $\frac{\delta}{\delta s}$, $\frac{\delta}{\delta n}$, $\frac{\delta}{\delta b}$ introduced subsequently will have similar meanings.

Write

$$\operatorname{curl} \mathbf{e}_{\alpha} = \pi_{\alpha\beta}\, \mathbf{e}_{\beta} \quad . \tag{1.5}$$

Then, since $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ the condition (1.1) gives

$$\pi_{12} = - \frac{\delta}{\delta e_3} \log v \tag{1.6}$$

and

$$\pi_{13} = \frac{\delta}{\delta e_2} \log v \quad . \tag{1.7}$$

One also has

$$\operatorname{div} \mathbf{e}_1 = \operatorname{div}(\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_3 \cdot \operatorname{curl} \mathbf{e}_2 - \mathbf{e}_2 \cdot \operatorname{curl} \mathbf{e}_3$$

$$= \pi_{23} - \pi_{32} \; , \tag{1.8}$$

and it follows from (1.3) that

$$\pi_{23} - \pi_{32} = - \frac{\delta}{\delta e_1} \log v \, . \tag{1.9}$$

Using (1.1), (1.4), (1.6) and (1.7) we obtain

$$\operatorname{curl} \boldsymbol{\omega} = v \left[\pi_{11}^2 \mathbf{e}_1 - \frac{\delta \pi_{11}}{\delta e_2} \mathbf{e}_3 \right] \quad . \tag{1.10}$$

To evaluate curl curl $\boldsymbol{\omega}$ from (1.10) one needs the relation

$$- \frac{\delta^2 \pi_{11}}{\delta e_1 \delta e_2} = \pi_{23} \frac{\delta \pi_{11}}{\delta e_2} \quad . \tag{1.11}$$

which may be obtained from the identity curl grad $\pi_{11} = 0$ together with (1.1), (1.4), and (1.5).

Using (1.1), (1.4), (1.6), (1.7), (1.9), (1.10) and (1.11) one obtains from the condition (1.8)

$$\pi_{23} \frac{\delta \pi_{11}}{\delta e_2} = 0 , \qquad (1.12)$$

$$(2\pi_{11} + \pi_{33}) \frac{\delta \pi_{11}}{\delta e_2} = 0 . \qquad (1.13)$$

Since $\frac{\delta \pi_{11}}{\delta e_2}$ cannot vanish, it follows from (1.12) and (1.13) that

$$\pi_{23} = 0 \qquad (1.14)$$

and

$$2\pi_{11} + \pi_{33} = 0 . \qquad (1.15)$$

It now follows from (1.3), (1.5) and (1.14) that

$$\text{curl } e_2 = \pi_{21} e_1 . \qquad (1.16)$$

The relation (1.16) asserts that the unit vector e_1, tangent to the stream line, points along the binormal to the vector-line of e_2. (The vector-lines of e_2 are the normals to the family of surfaces π_{11} = constant.)

2. PRELIMINARY ANALYSIS

One may now write $\underset{\sim}{s}$, $\underset{\sim}{n}$ and $\underset{\sim}{b}$ respectively in place of $\underset{\sim}{e}_2$, $\underset{\sim}{e}_3$ and $\underset{\sim}{e}_1$. The unit vector $\underset{\sim}{s}$ is normal to the surfaces

$$\Omega_b = \underset{\sim}{b}\cdot\text{curl}\,\underset{\sim}{b} = \text{constant} \;, \tag{2.1}$$

while $\underset{\sim}{n}$ and $\underset{\sim}{b}$ point along the principal normal and binormal to the vector-lines of $\underset{\sim}{s}$, and $\underset{\sim}{b}$ is tangent to the stream line. For the particular case when the s-lines are rectilinear (so that the surfaces (2.1) are parallel) one defines the binormal $\underset{\sim}{b}$ to be tangent to the stream line.

The formulae associated with this representation of the vector field are given in the Appendix to a paper by Marris & Wang (1970). In particular, the abnormalities of the vector-lines of $\underset{\sim}{s}$, $\underset{\sim}{n}$ and $\underset{\sim}{b}$ are related to the torsion τ of the $\underset{\sim}{s}$-lines by

$$\Omega_n + \Omega_b = \Omega_s - 2\tau \;. \tag{2.2}$$

It then follows from (1.3, (1.15) and (2.2) that

$$\Omega_s = 0 \;, \quad \Omega_n = -4\tau \;, \quad \Omega_b = 2\tau \;. \tag{2.3}$$

The surfaces orthogonal to the s-lines are surfaces upon which the torsion τ maintains a constant value.*

---

*In the new notation the case τ spatially constant corresponds to a Trkalian motion, while the case $\tau = 0$ gives irrotational motion. Both these cases are discounted throughout the present work.

The following relations hold for the vector field:

$$\operatorname{curl} \underset{\sim}{s} = \kappa\, \underset{\sim}{b} \ , \tag{2.4}$$

$$\operatorname{curl} \underset{\sim}{n} = -\operatorname{div} \underset{\sim}{b}\, \underset{\sim}{s} - 4\tau\, \underset{\sim}{n} + \psi\, \underset{\sim}{b} \ , \tag{2.5}$$

$$\operatorname{curl} \underset{\sim}{b} = (\kappa + \operatorname{div} \underset{\sim}{n})\underset{\sim}{s} - \theta\, \underset{\sim}{n} + 2\tau\, \underset{\sim}{b} \ , \tag{2.6}$$

where κ is the curvature of the $\underset{\sim}{s}$-lines.

Commutation formulae may be obtained by applying the identity $\operatorname{curl} \operatorname{grad} F = \underset{\sim}{0}$ to the tensor point function F, namely

$$\frac{\delta^2 F}{\delta b \delta n} - \frac{\delta^2 F}{\delta n \delta b} = -\operatorname{div} \underset{\sim}{b} \frac{\delta F}{\delta n} + (\kappa + \operatorname{div} \underset{\sim}{n}) \frac{\delta F}{\delta b} \ , \tag{2.7}$$

$$\frac{\delta^2 F}{\delta s \delta b} - \frac{\delta^2 F}{\delta b \delta s} = -4\tau \frac{\delta F}{\delta n} - \theta \frac{\delta F}{\delta b} \ , \tag{2.8}$$

$$\frac{\delta^2 F}{\delta n \delta s} - \frac{\delta^2 F}{\delta s \delta n} = \kappa \frac{\delta F}{\delta s} + \psi \frac{\delta F}{\delta n} + 2\tau \frac{\delta F}{\delta b} \ . \tag{2.9}$$

When these formulae are applied to the basis vectors $\underset{\sim}{s}$, $\underset{\sim}{n}$ and $\underset{\sim}{b}$ we obtain nine compatibility conditions. With the simplification afforded by (2.3), and also by the conditions

$$\frac{\delta \tau}{\delta n} = 0 \ , \quad \frac{\delta \tau}{\delta b} = 0 \ , \tag{2.10}$$

one obtains the following relations,

$$\frac{\delta \kappa}{\delta b} + \kappa \operatorname{div} \underset{\sim}{b} = 0 \ . \tag{2.11}$$

$$\frac{\delta \psi}{\delta b} - \theta\tau\, (\kappa + \operatorname{div} \underset{\sim}{n}) \ldots (\psi - \theta)\operatorname{div} \underset{\sim}{b} = 0 \ , \tag{2.12}$$

$$\frac{\delta\theta}{\delta n} - 6\tau \operatorname{div} \underset{\sim}{b} + (\theta-\psi)(\kappa + \operatorname{div} \underset{\sim}{n}) = 0 \;, \tag{2.13}$$

$$3\, \frac{\delta\tau}{\delta s} + \kappa \operatorname{div} \underset{\sim}{b} + 2\tau(\theta + 2\psi) = 0 \;, \tag{2.14}$$

$$\frac{\delta}{\delta s}(\kappa + \operatorname{div} \underset{\sim}{n}) - 4\tau \operatorname{div} \underset{\sim}{b} + \theta(2\kappa + \operatorname{div} \underset{\sim}{n}) = 0 \;, \tag{2.15}$$

$$\frac{\delta\theta}{\delta s} + \theta^2 - \kappa(\kappa + \operatorname{div} \underset{\sim}{n}) + 15\tau^2 = 0 \;, \tag{2.16}$$

$$\frac{\delta\kappa}{\delta n} - \frac{\delta\psi}{\delta s} - \kappa^2 - \psi^2 - 3\tau^2 = 0 \;, \tag{2.17}$$

$$\frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b} - 4\tau\kappa + \psi \operatorname{div} \underset{\sim}{b} - 2\tau(\kappa + \operatorname{div} \underset{\sim}{n}) = 0 \;, \tag{2.18}$$

$$\begin{aligned} \frac{\delta}{\delta n}(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} + \theta\psi + (\operatorname{div} \underset{\sim}{b})^2 & \\ + (\kappa + \operatorname{div} \underset{\sim}{n})^2 - 9\tau^2 = 0 \;. & \end{aligned} \tag{2.19}$$

The condition (1.1) now requires that

$$\kappa + \operatorname{div} n = - \frac{\delta}{\delta n} \log v \;, \tag{2.20}$$

$$\theta = - \frac{\delta}{\delta s} \log v \;, \tag{2.21}$$

these relations being evidently equivalent to (1.6) and (1.7). From (1.3) one has

$$\operatorname{div} \underset{\sim}{b} = - \frac{\delta}{\delta b} \log v \; . \tag{2.22}$$

From (2.20), (2.21) and (2.22) and the commutation formula (2.8) one has

$$\frac{\delta\theta}{\delta b} = - \frac{\delta^2}{\delta b \delta s} \log v = - \frac{\delta^2}{\delta s \delta b} \log v - 4\tau \frac{\delta}{\delta n} \log v - \theta \frac{\delta}{\delta b} \log v$$

$$= \frac{\delta}{\delta s} \operatorname{div} \underset{\sim}{b} + 4\tau(\kappa + \operatorname{div} \underset{\sim}{n}) + \theta \operatorname{div} \underset{\sim}{b}. \tag{2.23}$$

By (2.18) and (2.23) one has also

$$\frac{\delta\theta}{\delta b} - 4\tau\kappa + (\psi - \theta)\operatorname{div} \underset{\sim}{b} - 6\tau(\kappa + \operatorname{div} \underset{\sim}{n}) = 0 \; , \tag{2.24}$$

and (2.12) and (2.24) together exhibit the interesting condition

$$\frac{\delta}{\delta b} (\theta - \psi) = 4\tau\kappa \; . \tag{2.25}$$

Taking the directional derivative of both sides of (2.20) with respect to $\underset{\sim}{b}$, and of (2.22) with respect to n, one immediately obtains from (2.7) the relation

$$\frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} - \frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) = 0 \; . \tag{2.26}$$

Applying the commutation formula (2.8) to τ, we obtain from (2.10)

$$\frac{\delta^2 \tau}{\delta b \delta s} = 0 \; . \tag{2.27}$$

Similarly from (2.9) and (2.10) we have

$$\frac{\delta^2\tau}{\delta n \delta s} = \kappa \frac{\delta\tau}{\delta s} \ . \tag{2.28}$$

Taking the directional derivative of (2.14) with respect to $\underset{\sim}{b}$, and then using (2.27) together with (2.11), (2.12) and (2.24), one obtains for $\frac{\delta}{\delta b}$ div $\underset{\sim}{b}$ the expression

$$\kappa \frac{\delta}{\delta b} \text{div } \underset{\sim}{b} - \kappa(\text{div } \underset{\sim}{b})^2 + 2\tau[4\tau\kappa + 3(\theta-\psi)\text{div } \underset{\sim}{b}$$

$$+ 18\tau(\kappa + \text{div } \underset{\sim}{n})] = 0 \ . \tag{2.29}$$

From (2.19) and (2.29) we obtain

$$\kappa \frac{\delta}{\delta n} (\kappa + \text{div } \underset{\sim}{n}) = - 2\kappa(\text{div } \underset{\sim}{b})^2 - \kappa(\kappa + \text{div } \underset{\sim}{n})^2 - \kappa(\theta\psi - 17\tau^2)$$

$$+ 2\tau[3(\theta-\psi)\text{div } \underset{\sim}{b} + 18\tau(\kappa + \text{div } \underset{\sim}{n})] \ . \tag{2.30}$$

The following formula, obtained from (2.16) and (2.17), will be used repeatedly

$$\frac{\delta}{\delta s} (\psi-\theta) = \frac{\delta\kappa}{\delta n} - \psi^2 + \theta^2 + 12\tau^2 - \kappa(\kappa + \text{div } \underset{\sim}{n}) - \kappa^2 \ . \tag{2.31}$$

We note from (2.11) that for $\kappa \neq 0$

$$\frac{\delta}{\delta b}\left[\frac{1}{\kappa^2}\frac{\delta\kappa}{\delta n}+\frac{\kappa+\mathrm{div}\,\mathbf{n}}{\kappa}\right] = \frac{2}{\kappa^2}\,\mathrm{div}\,\mathbf{b}\,\frac{\delta\kappa}{\delta n}+\frac{1}{\kappa^2}\frac{\delta^2\kappa}{\delta b\delta n}$$

$$+\frac{\mathrm{div}\,\mathbf{b}(\kappa+\mathrm{div}\,\mathbf{n})}{\kappa}+\frac{1}{\kappa}\frac{\delta}{\delta b}(\kappa+\mathrm{div}\,\mathbf{n}) \quad .$$

By (2.7) and (2.11)

$$\frac{\delta^2\kappa}{\delta b\delta n} = \frac{\delta}{\delta n}(-\kappa\,\mathrm{div}\,\mathbf{b}) - \mathrm{div}\,\mathbf{b}\frac{\delta\kappa}{\delta n} - \kappa(\kappa+\mathrm{div}\,\mathbf{n})\mathrm{div}\,\mathbf{b}$$

$$= -2\,\mathrm{div}\,\mathbf{b}\frac{\delta\kappa}{\delta n} - \kappa\left[\frac{\delta}{\delta n}\mathrm{div}\,\mathbf{b} + (\kappa+\mathrm{div}\,\mathbf{n})\mathrm{div}\,\mathbf{b}\right] \quad ,$$

and it follows from (2.26) that

$$\frac{\delta}{\delta b}\left[\frac{1}{\kappa^2}\frac{\delta\kappa}{\delta n}+\frac{\kappa+\mathrm{div}\,\mathbf{n}}{\kappa}\right] = 0 \quad . \tag{2.32}$$

The condition (2.32) may be written

$$\frac{\delta\kappa}{\delta n}+\kappa(\kappa+\mathrm{div}\,\mathbf{n}) = \mu\kappa^2 \tag{2.33}$$

where μ is a parameter bearing a constant value on a $\mathbf{b}$-line

$$\frac{\delta\mu}{\delta b} = 0 \quad . \tag{2.34}$$

From (2.31) and (2.33) we evidently have

$$\frac{1}{\kappa^2}\left[\frac{\delta}{\delta s}(\psi-\theta) + 2\kappa(\kappa + \operatorname{div} \underset{\sim}{n}) + \psi^2 + \zeta^2 - \theta^2 - 12\tau^2\right] = \mu, \tag{2.35}$$

Again from (2.14) we may write

$$\kappa \operatorname{div} \underset{\sim}{b} + 2\tau(\theta + 2\psi) = -3\frac{\delta\tau}{\delta s} = \lambda, \tag{2.36}$$

and it follows from (2.27) that λ bears a constant value on a $\underset{\sim}{b}$-line,

$$\frac{\delta\lambda}{\delta b} = 0 \quad . \tag{2.37}$$

Finally we write

$$\zeta^2 = (\theta-\psi)^2 + 36\tau^2 \tag{2.38}$$

and it follows from (2.10) and (2.25) that

$$\frac{\delta\zeta^2}{\delta b} = 8\tau(\theta-\psi) \quad . \tag{2.39}$$

3. THE CASE $\kappa = 0$

We remember that the vector-lines of $\underset{\sim}{s}$ are the orthogonal trajectories of the surfaces on which the stream line abnormality Ω_b (or τ) bears a constant value. A special case arises when the vector-lines of $\underset{\sim}{s}$ are rectilinear so that these surfaces are parallel surfaces. We obtain a motion whose steady stream-lines are circular helices. A rather careful analysis is required to show that this motion is impossible for Navier-Stokes fluids.

When $\kappa = 0$ it follows from (2.29), since $\tau \neq 0$, that

$$(\theta-\psi)\mathrm{div}\,\underset{\sim}{b} + \sigma\,\mathrm{div}\,\underset{\sim}{n} = 0 \tag{3.1}$$

and by (2.12) and (3.1)

$$\frac{\delta\psi}{\delta b} = 0 \ . \tag{3.2}$$

By (2.17)

$$\frac{\delta\psi}{\delta s} = -\psi^2 - 3\tau^2 \ , \tag{3.3}$$

hence by (2.10)^2, (3.2) and (3.3)

$$\frac{\delta^2\psi}{\delta b\,\delta s} = 0 \ . \tag{3.4}$$

By (3.2), $\frac{\delta^2\psi}{\delta s\,\delta b} = 0$, so on applying the commutation formula (2.8) to ψ, and using (3.2), we obtain

$$\tau\frac{\delta\psi}{\delta n} = 0 \ . \tag{3.5}$$

By (2.3)[3] the vanishing of ε implies $\Omega_b = 0$, which is the case of irrotational motion which is discounted. We therefore have

$$\frac{\delta\psi}{\delta n} = 0 \; . \tag{3.6}$$

Taking the directional derivative of (2.14) with respect to $\underset{\sim}{n}$ using (2.10) and (2.28) we have for $\kappa = 0$,

$$2\varepsilon\left(\frac{\delta\theta}{\delta n} + 2\,\frac{\delta\psi}{\delta n}\right) = 0 \; , \tag{3.7}$$

and since ε is non-vanishing we conclude from (3.6) and (3.7) that

$$\frac{\delta\theta}{\delta n} = 0 \; , \tag{3.7}$$

and from (2.13) and (3.7)

$$\theta\,\mathrm{div}\,\underset{\sim}{b} \quad (\theta-\psi)\,\mathrm{div}\,\underset{\sim}{n} = 0 \tag{3.8}$$

For (3.1) and (3.8) to give non-zero values of div $\underset{\sim}{b}$ and div $\underset{\sim}{n}$ one must have

$$\varepsilon^2 = (\theta-\psi)^2 + 3\theta\psi^2 = 0 \tag{3.9}$$

which is impossible for non-vanishing ε. We conclude that div $\underset{\sim}{b}$ and div $\underset{\sim}{n}$ must be zero. It appears from (2.10), (2.20), (2.22), (2.24), (2.25), (3.6) and (3.7), that θ, ψ and ε and also the velocity magnitude v are constant on the surfaces orthogonal to the vector lines of $\underset{\sim}{s}$. The conditions (2.4), (2.5) and (2.6) reduce to

$$\text{curl}\ \underset{\sim}{s} = 0 \ , \tag{3.10}$$

$$\text{curl}\ \underset{\sim}{n} = -4\tau\, \underset{\sim}{n} + \psi\, \underset{\sim}{b} = \Omega_n \underset{\sim}{n} + \psi\, \underset{\sim}{b} \ , \tag{3.11}$$

$$\text{curl}\ \underset{\sim}{b} = -\theta\, \underset{\sim}{n} + 2\tau\, \underset{\sim}{b} = -\theta\, \underset{\sim}{n} + \Omega_b\, \underset{\sim}{b} \ . \tag{3.12}$$

Writing (3.12) in the form

$$\text{curl}\ \underset{\sim}{b} = \Omega_b\, \underset{\sim}{b} + \kappa_b\, \underset{\sim}{b}_b \ , \tag{3.13}$$

where κ_b is the curvature of the $\underset{\sim}{b}$-line and $\underset{\sim}{b}_b$ is the unit bi-normal to the $\underset{\sim}{b}$-line, we have

$$\kappa_b = -\theta \ , \tag{3.14}$$

and

$$\underset{\sim}{b}_b = \underset{\sim}{n} \ . \tag{3.15}$$

The principal normal $\underset{\sim}{n}_b$ to the $\underset{\sim}{b}$-line is $\underset{\sim}{b}_b \times \underset{\sim}{b} = \underset{\sim}{n} \times \underset{\sim}{b} = \underset{\sim}{s}$. Thus the $\underset{\sim}{b}$-lines are geodesics on the surfaces whose normal is $\underset{\sim}{s}$. The torsion τ_b of the $\underset{\sim}{b}$-lines is given by

$$\underset{\sim}{b} \cdot \text{grad}\ \underset{\sim}{b}_b = -\tau_b\, \underset{\sim}{n}_b$$

or

$$-\tau_b = \underset{\sim}{b} \cdot \text{grad}\ \underset{\sim}{n} \cdot \underset{\sim}{s} \qquad \text{by (3.15)} \ .$$

Using the expansion for grad $\underset{\sim}{n}$ (see (1970), Appendix), we obtain

$$-\tau_b = \Omega_n + \tau = -3\tau \qquad \text{by (2.3)} \ .$$

Thus

$$\tau_b = 3\tau \quad . \tag{3.16}$$

The vector-lines of $\underset{\sim}{b}$ (stream lines) are curves of constant curvature and torsion, they are circular helices.

Equation (2.19), which is the Gauss equation for the surfaces orthogonal to the vector lines of $\underset{\sim}{s}$, reduces to

$$\theta\psi - 9\tau^2 = 0 \ . \tag{3.17}$$

Equation (3.17) asserts that the Gaussian' curvature of these surfaces is zero. The first curvature of the surfaces, given by $-(\theta + \psi)$, is constant over a representative surface. The surfaces orthgonal to $\underset{\sim}{s}$ must be concentric circular cylinders.*

We write

$$\tan \alpha = \frac{\kappa_b}{\tau_b} = -\frac{\theta}{3\tau} = -\frac{\theta}{\tau_b} \tag{3.18}$$

so that α is the angle of inclination of the stream-lines to the generators of the cylinders. From (3.17) and (3.18),

$$\cot \alpha = -\frac{\psi}{3\tau} = -\frac{\psi}{\tau_b} \quad . \tag{3.19}$$

---

*A theorem of geometry asserts that a family of surfaces having constant mean and Gaussian curvature and whose orthogonal trajectories are straight lines must consist of concentric spheres, parallel planes or concentric circular cylinders. A proof of this was given by Ericksen (1954, p. 475 f). In the present case the surfaces are developables. The family must consist of parallel planes or concentric cylinders. But if they were planes we would have $\tau = 0$, and this case is discounted. The surfaces must be concentric circular cylinders.

We write $\frac{\delta}{\delta s} = -\frac{\partial}{\partial r}$ where r is the radius of the representative cylinder. The conditions (2.14), (2.16), (2.17), and (2.21) reduce respectively to

$$\frac{\partial \tau_b}{\partial r} = \frac{2}{3}\tau_b\ (\theta + 2\psi) = \frac{2}{3}\tau_b^2(\tan\alpha + 2\cot\alpha)\ , \tag{3.20}$$

$$\frac{\partial \theta}{\partial r} = \theta^2 + 15\tau^2 = \tau_b^2\left(\tan^2\alpha + \frac{5}{3}\right)\ , \tag{3.21}$$

$$\frac{\partial \psi}{\partial r} = \psi^2 + 3\tau^2 = \tau_b^2\left(\cot^2\alpha + \frac{1}{3}\right)\ , \tag{3.22}$$

$$\frac{\partial}{\partial r}\log v = \theta\ . \tag{3.23}$$

From (3.17), (3.21)[1] and (3.22)[1] we get

$$\frac{\partial}{\partial r}(\theta+\psi) = (\theta+\psi)^2$$

which integrates to

$$\theta + \psi = -\frac{1}{r} \tag{3.24}$$

where the constant of integration is absorbed in the choice of origin for r. Since $\theta + \psi = \operatorname{div} \underset{\sim}{s}$, equation (3.24) verifies that the first curvature of the surface is $-\operatorname{div} \underset{\sim}{s}$.

From (3.18), (3.19) and (3.24) we get

$$3\tau\ (\cot\alpha + \tan\alpha) = \frac{1}{r}\ ,$$

or

$$\tau_b = 3\tau = \frac{\sin 2\alpha}{2r} \quad , \tag{3.25}$$

and

$$\theta = -\frac{\sin^2\alpha}{r} \quad , \tag{3.26}$$

$$\psi = -\frac{\cos^2\alpha}{r} \quad . \tag{3.27}$$

From (3.19) and (3.22)[2]

$$\frac{\partial\psi}{\partial r} = -\frac{\partial\tau_b}{\partial r}\cot\alpha + \tau_b \operatorname{cosec}^2\alpha \frac{\partial\alpha}{\partial r} - \tau_b^2\left(\cot^2\alpha + \frac{1}{3}\right)$$

or, substituting for $\frac{\partial\tau_b}{\partial r}$ from (3.20) and cancelling τ_b, we obtain

$$\frac{\partial\alpha}{\partial r} = -\frac{\tau_b}{3} = -\tau = -\frac{\sin 2\alpha}{6r} \quad . \tag{3.28}$$

We may verify that the same result follows from (3.18) and (3.21)[2].

Since the stream lines are steady the curvature θ given by (3.26) is not a function of time. We conclude from (3.23) that

$$\frac{\partial^2}{\partial r \partial t} \log v = 0 \quad . \tag{3.29}$$

The vorticity field must be a solution to the three scalar equations of motion (1.4). We have satisfied the two conditions given by (1.8), but we have yet to satisfy

$$\mathbf{b} \cdot \frac{\partial\omega}{\partial t} = \nu\mathbf{b} \cdot \operatorname{curl}\operatorname{curl}\omega \quad . \tag{3.30}$$

We have

$$\underset{\sim}{\omega} = \delta_b \underset{\sim}{v} = 2\tau v \underset{\sim}{b} , \quad \text{by } (2.5) \tag{3.31}$$

so that

$$\underset{\sim}{b} \cdot \frac{\partial \underset{\sim}{\omega}}{\partial t} = 2\tau \frac{\partial v}{\partial t} = \frac{\sin 2\alpha}{3r} \frac{\partial v}{\partial t} , \quad \text{by } (3.25) . \tag{3.32}$$

Using (2.4), (2.5), (2.6), (2.20), (2.21) and (2.22)

$$\text{curl } \underset{\sim}{\omega} = v \left[4\tau^2 \underset{\sim}{b} - 2 \frac{\partial \tau}{\partial s} \underset{\sim}{n} \right] = v \left[4\tau^2 \underset{\sim}{b} + 2 \frac{\partial \tau}{\partial r} \underset{\sim}{n} \right] , \tag{3.33}$$

and

$$\text{curl curl } \underset{\sim}{\omega} = v \left[2 (\theta - \phi) \frac{\partial \tau}{\partial s} - 2 \frac{\partial^2 \tau}{\partial s^2} + 8\tau^3 \right] \underset{\sim}{b}$$

$$= v \left[- 2(\theta - \phi) \frac{\partial \tau}{\partial r} - 2 \frac{\partial^2 \tau}{\partial r^2} + 8\tau^3 \right] \underset{\sim}{b} . \tag{3.34}$$

From (3.25) and (3.28)

$$\frac{\partial \tau}{\partial r} = \frac{\partial}{\partial r} \left[\frac{\sin 2\alpha}{6r} \right] = \frac{\sin 2\alpha}{18r^2} [\cos 2\alpha + 3] , \tag{3.35}$$

and

$$\frac{\partial^2 \tau}{\partial r^2} = \frac{1}{r^3} \left[\frac{\sin 4\alpha}{12} + \frac{\sin 2\alpha}{3} + \frac{\cos 4\alpha \sin 2\alpha}{54} \right] . \tag{3.36}$$

From (3.26) and (3.27)

$$\theta = \psi = \frac{\cos 2\alpha}{r} \tag{3.37}$$

Substituting these expressions into (3.34) we get

$$\text{curl curl } \underset{\sim}{\omega} = -\frac{16}{27}\frac{\nu}{r^3}\sin 3\alpha\, \underset{\sim}{b} . \tag{3.38}$$

Substituting the expressions $(3.32)^2$ and (3.38) into (3.30), we get

$$\frac{\partial}{\partial t}\log \nu = \frac{16}{9}\frac{\nu}{r^2} . \tag{3.39}$$

The conditions (3.29) and (3.39) are incompatible unless $\nu = 0$. The motion is thus impossible for Navier-Stokes fluids.

4. FURTHER ANALYSIS. A BASIC INTEGRAL.

The case $\kappa = 0$ having been dealt with we shall hence forth require that $\kappa \neq 0$. We are reminded also that τ and $\frac{\delta\tau}{\delta s}$ are non-vanishing to discount irrotational motions and Trkalian motions respectively.

Taking the gradient of (2.14) with respect to $\underset{\sim}{n}$ using (2.10)[1], (2.13) and (2.28), and replacing $\frac{\delta\kappa}{\delta n}$ by $\frac{\delta}{\delta s}$ $(\psi-\theta)$ by (2.31) we obtain on reduction*

$$\operatorname{div} \underset{\sim}{b} \frac{\delta}{\delta s} (\psi-\theta) + 4\tau \frac{\delta}{\delta n} (\psi-\theta) + \kappa\left[\frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} + (\kappa + \operatorname{div} \underset{\sim}{n})\operatorname{div} \underset{\sim}{b}\right.$$

$$\left. + \operatorname{div} \underset{\sim}{b}(\psi^2-\theta^2) + 4\tau^2\right] - 2\tau\kappa(\theta + 2\psi)$$

$$- 6\tau(\theta-\psi)(\kappa + \operatorname{div} \underset{\sim}{n}) = 0 . \tag{4.1}$$

We may now take the gradient of (2.26) with respect to $\underset{\sim}{s}$. We use the commutation formulae (2.8) and (2.9), we use (2.15) and (2.18) to eliminate $\frac{\delta}{\delta s}$ $(\kappa + \operatorname{div} \underset{\sim}{n})$ and $\frac{\delta}{\delta s}$ $\operatorname{div} \underset{\sim}{b}$, we use (2.31) to eliminate $\frac{\delta\kappa}{\delta n}$ in favour of $\frac{\delta\psi}{\delta s}$, and finally we use (2.30) to eliminate $\frac{\delta}{\delta n}$ $(\kappa + \operatorname{div} \underset{\sim}{n})$.

*The formula (4.1) may be obtained alternatively by writing (2.26) as

$$\frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) = - \frac{\delta^2}{\delta n \delta b} \log \kappa = - \frac{\delta^2}{\delta b \delta n} \log \kappa + \ldots$$

We then use (2.17) to replace $\frac{\delta\tau}{\delta n}$ by $\frac{\delta\psi}{\delta s}$, and replace $\frac{\delta^2\psi}{\delta b\delta s}$ by $\frac{\delta^2\psi}{\delta s\delta b}$ using the commutation formula (2.8). On reduction we check (4.1).

On reduction we obtain*

$$
\begin{aligned}
&4\tau \frac{\delta}{\delta s}(\psi-\theta) - \operatorname{div}\mathbf{b}\,\frac{\delta}{\delta n}(\psi-\theta) + 2(\theta-\psi)\left[\frac{\delta}{\delta n}\operatorname{div}\mathbf{b} + (\kappa + \operatorname{div}\mathbf{n})\operatorname{div}\mathbf{b}\right] \\
&\quad + \tau\Big\{-24(\operatorname{div}\mathbf{b})^2 + 12\kappa(\kappa + \operatorname{div}\mathbf{n}) + 102\tau^2 + 4\kappa^2 \\
&\quad + 4(\psi^2-\theta^2) - 68\theta\psi + \frac{72\tau}{\kappa}\left[(\theta-\psi)\operatorname{div}\mathbf{b} + 6\tau(\kappa + \operatorname{div}\mathbf{n})\right]\Big\} = 0 \ . \qquad (4.2)
\end{aligned}
$$

We may eliminate $\frac{\delta}{\delta s}(\psi-\theta)$ from (4.2) in favour of $\frac{\delta\kappa}{\delta n}$ by (2.31). We may see (4.2) as an expression for the product $\theta\psi\tau$ in terms $\frac{\delta\kappa}{\delta n}$, $\frac{\delta}{\delta n}(\psi-\tau)$, $\frac{\delta}{\delta n}(\operatorname{div}\mathbf{b})$, $\theta-\psi$, $\kappa + \operatorname{div}\mathbf{n}$, κ, τ and $\operatorname{div}\mathbf{b}$, thus

$$
\begin{aligned}
60\theta\psi\tau = {} & 4\tau\frac{\delta\kappa}{\delta n} - \operatorname{div}\mathbf{b}\,\frac{\delta}{\delta n}(\psi-\theta) \\
&+ 2(\theta-\psi)\left[\frac{\delta}{\delta n}\operatorname{div}\mathbf{b} + (\kappa + \operatorname{div}\mathbf{n})\operatorname{div}\mathbf{b}\right] \\
&+ \tau\Big[-24(\operatorname{div}\mathbf{b})^2 + 8\kappa(\kappa + \operatorname{div}\mathbf{n}) + 150\tau^2 \\
&+ \frac{72\tau}{\kappa}\left[(\theta-\psi)\operatorname{div}\mathbf{b} + 6\tau(\kappa + \operatorname{div}\mathbf{n})\right]\Big] \ . \qquad (4.3)
\end{aligned}
$$

*The formula (4.2) may be obtained alternatively as follows. From (2.12) we may obtain an expression for $\frac{\delta^2\psi}{\delta n\delta b}$. If we take the gradient of (2.15) with respect to $\mathbf{b}$ we obtain an expression for $\frac{\delta^2\theta}{\delta b\delta n}$. Using the commutation formula (2.7) applied to $\mathbf{n}$ we then get an expression for $\frac{\delta^2\theta}{\delta n\delta b}$. By addition we get an expression for $\frac{\delta^2}{\delta n\delta b}(\theta+\psi)$. Also from (2.12) and (2.24) we have an expression for $\frac{\delta}{\delta b}(\theta+\psi)$. Taking the gradient of this last expression with respect to $\mathbf{n}$ and using the previous expression for $\frac{\delta^2}{\delta n\delta b}(\theta+\psi)$ we obtain (4.2). The condition (4.2) may also be derived by starting by taking the derivative of (2.18) with respect to $\mathbf{n}$. Thus we have checks on (4.2).

A third condition of this order is obtained by taking the directional derivative of (2.24) with respect to $\underset{\sim}{s}$. We use the commutation formula (2.8) applied to θ, and apply the formulae (2.11) to (2.19), (2.24) and (2.26) as required. On reduction we obtain

$$-\operatorname{div}\underset{\sim}{b}\,\frac{\delta}{\delta s}(\psi-\theta) - \kappa\left[\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + (\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b}\right] + 4\tau\left[\frac{\delta\kappa}{\delta s} + \theta\kappa\right]$$
$$+\left[-\frac{4\kappa^2}{3} + (\psi-3\theta)(\psi-\theta) + 48\tau^2\right]\operatorname{div}\underset{\sim}{b} + 6\tau(\theta-\psi)(\kappa + \operatorname{div}\underset{\sim}{n})$$
$$+\frac{(10\theta - 28\psi)}{3}\kappa\tau = 0 . \qquad (4.4)$$

We now subtract (4.4) from (4.1). We obtain*

$$2\operatorname{div}\underset{\sim}{b}\,\frac{\delta}{\delta s}(\psi-\theta) + 4\tau\left[\frac{\delta}{\delta n}(\psi-\theta) - \left(\frac{\delta\tau}{\delta s} + \theta\kappa\right)\right] + 2\kappa\left[\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + (\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b}\right]$$
$$+\operatorname{div}\underset{\sim}{b}\left[4\theta(\psi-\theta) - 24\tau^2 + \frac{4\kappa^2}{3}\right] - 12\tau(\theta-\psi)(\kappa + \operatorname{div}\underset{\sim}{n})$$
$$-\frac{16}{3}\tau\kappa(\theta-\psi) = 0 .$$

If now we eliminate $\frac{\delta}{\delta s}(\psi-\theta)$ in favour of $\frac{\delta\kappa}{\delta n}$ using (2.31) we obtain

$$2\operatorname{div}\underset{\sim}{b}\,\frac{\delta\kappa}{\delta n} + 2\kappa\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + 4\tau\left[\frac{\delta}{\delta n}(\psi-\theta) - \left(\frac{\delta\tau}{\delta s} + \theta\kappa\right)\right] - 2\left(\frac{\kappa^2}{3} + (\theta-\psi)^2\right)\operatorname{div}\underset{\sim}{b}$$
$$-12\tau(\theta-\psi)(\kappa + \operatorname{div}\underset{\sim}{n}) - \frac{16}{3}\tau\kappa(\theta-\psi) = 0 . \qquad (4.5)$$

*The condition (4.5) may also be obtained by taking the directional derivative of (2.35) with respect to $\underset{\sim}{b}$ and using (2.34). The gradient $\frac{\delta}{\delta s}(\psi-\theta)$ is expressed in terms of $\frac{\delta\kappa}{\delta n}$ by (2.31). This serves as a check on (4.1) and (4.4).

The condition (4.5) is significant in that except for the term $\frac{\delta\kappa}{\delta s} + \theta\kappa$ which will eventually be eliminated, it contains θ and ψ as the single variable θ-ψ.

Adding (4.1) and (4.4) we obtain*

$$4\tau\left[\frac{\delta\kappa}{\delta s} + \theta\kappa + \frac{\delta}{\delta n}(\psi-\theta)\right] + F = 0\ , \qquad (4.6)$$

where

$$F = \operatorname{div}\underset{\sim}{b}\left[2\xi^2 - \frac{4\kappa^2}{3}\right] + \frac{4\tau\kappa}{3}\ (\theta - 10\psi)\ , \qquad (4.7)$$

and ξ^2 is given by (2.38). This condition will subsequently be used to express ψ in terms of $\frac{\delta\kappa}{\delta s} + \theta\kappa$, $\frac{\delta}{\delta n}(\psi-\theta)$, θ-ψ, κ, τ and div $\underset{\sim}{b}$. We have

$$\tau\kappa\psi = \frac{1}{3}\left[\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) + \frac{\delta}{\delta n}(\psi-\theta)\right] + \frac{\operatorname{div} b}{12}\left[2\xi^2 - \frac{4\kappa^2}{3}\right]$$

$$+ \frac{\tau\kappa}{9}\ (\theta-\psi)\ . \qquad (4.8)$$

---

*The condition (4.6), (4.7) may be obtained by an alternative more circuitous route as follows. From (2.16) and (2.17) we have

$$\frac{\delta}{\delta s}(\theta+\psi) = \frac{\delta\kappa}{\delta n} - \kappa^2 + \kappa(\kappa + \operatorname{div}\underset{\sim}{n}) - \theta^2 - \psi^2 - 18\tau^2\ , \quad a)$$

while from (2.12) and (2.24)

$$\frac{\delta}{\delta b}(\theta+\psi) = 4\tau\kappa - 2(\psi-\theta)\operatorname{div} b + 12\tau(\kappa + \operatorname{div}\underset{\sim}{n})\ . \qquad b)$$

We can obtain the condition by applying the commutation formula (2.8) to $\theta+\psi$, using the above expressions and reducing using the formulae of Chapter 2. This serves as a further check on (4.1) and (4.4).

A polynomial integral among the variables ξ^2, div $\underset{\sim}{b}$, θ-ψ, κ, τ, and κ + div $\underset{\sim}{n}$ may now be obtained by taking the directional derivative of (4.7) with respect to $\underset{\sim}{b}$.

We have

$$\frac{\delta}{\delta b}\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) = \frac{\delta^2\kappa}{\delta b\delta s} + \kappa\,\frac{\delta\theta}{\delta b} - \theta\kappa \text{ div } \underset{\sim}{b}$$

$$= \frac{\delta^2\kappa}{\delta s\delta b} + 4\tau\,\frac{\delta\kappa}{\delta n} + \theta\,\frac{\delta\kappa}{\delta b} + \kappa\,\frac{\delta\theta}{\delta b} - \theta\kappa \text{ div } \underset{\sim}{b}\text{ , by (2.8).}$$

Substituting for $\frac{\delta\kappa}{\delta b}$ from (2.11) and then using (2.23) to eliminate $\frac{\delta\theta}{\delta b} - \frac{\delta}{\delta s}$ div $\underset{\sim}{b}$ we obtain

$$\frac{\delta}{\delta b}\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) = -\text{ div } \underset{\sim}{b}\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) + 4\tau\left(\frac{\delta\kappa}{\delta n} + \kappa(\kappa + \text{div } \underset{\sim}{n})\right) . \tag{4.9}$$

Again by (2.7), (2.10) and (2.25)

$$\frac{\delta^2}{\delta b\delta n}(\psi-\theta) = -4\tau\left(\frac{\delta\kappa}{\delta n} + \kappa(\kappa + \text{div } \underset{\sim}{n})\right) - \text{div } \underset{\sim}{b}\,\frac{\delta}{\delta n}(\psi-\theta) . \tag{4.10}$$

From (4.9) and (4.10)

$$\frac{\delta}{\delta b}\left(\frac{\delta\kappa}{\delta s} + \theta\kappa + \frac{\delta}{\delta n}(\psi-\theta)\right) + \text{div } b\left(\frac{\delta\kappa}{\delta s} + \theta\kappa + \frac{\delta}{\delta n}(\psi-\theta)\right) = 0 \tag{4.11}$$

and hence by (2.10) and (4.7)

$$\frac{\delta F}{\delta b} + \text{div } b\, F = 0 . \tag{4.12}$$

Using (4.7), (2.10), (2.11), (2.12), (2.24), (2.25), (2.29) and (2.39) we obtain

$$
\begin{aligned}
\frac{\delta F}{\delta \mathbf{b}} = \Bigg[2\xi^2 - \frac{4\kappa^2}{3}\Bigg]\Bigg[(\operatorname{div} \mathbf{b})^2 - 8\tau^2 \\
- \frac{2\tau}{\kappa}\,[3(\theta-\psi)\operatorname{div} \mathbf{b} + 18\tau(\kappa + \operatorname{div} \mathbf{n})]\Bigg] \\
+ \frac{8}{3}\,(\kappa \operatorname{div} \mathbf{b})^2 + \operatorname{div} \mathbf{b}\,\Bigg[16\tau\kappa(\theta-\psi) - \frac{4}{3}\kappa\tau(\theta - 10\psi)\Bigg] \\
+ \kappa\tau\Bigg[\frac{16}{3}\tau\kappa + 12[(\psi-\theta)\operatorname{div} \mathbf{b} - 6\tau(\kappa + \operatorname{div} \mathbf{n})]\Bigg],
\end{aligned}
\tag{4.13}
$$

and finally from (4.7), (4.12) and (4.13) we obtain, on reduction

$$
\begin{aligned}
\xi^2\,(\operatorname{div} \mathbf{b})^2 - (\xi^2 + \kappa^2)\,\tau[4\tau\kappa + 3(\theta-\psi)\operatorname{div} \mathbf{b}] \\
- 6[3\xi^2 + \kappa^2]\,\tau^2(\kappa + \operatorname{div} \mathbf{n}) = 0\ .
\end{aligned}
\tag{4.14}
$$

This basic relation expresses $(\kappa + \operatorname{div} \mathbf{n})$ explicitly in terms of the parameters $\theta-\psi$, $\kappa\tau$ and $\operatorname{div} \mathbf{b}$.

We may now take the directional derivative of (4.14) with respect to $\mathbf{b}$ and obtain an explicit formula for $\frac{\delta}{\delta \mathbf{b}}\,(\kappa + \operatorname{div} \mathbf{n})$. We have

$$
\begin{aligned}
\Big[\kappa(\operatorname{div} \mathbf{b})^2 - \tau(4\tau\kappa + 3(\theta-\psi)\operatorname{div} \mathbf{b}) - 18\tau^2(\kappa + \operatorname{div} \mathbf{n})\Big]\,\frac{\delta\xi^2}{\delta \mathbf{b}} \\
+ \Big[\xi^2(\operatorname{div} \mathbf{b})^2 + 2\kappa\tau(4\tau\kappa + 3(\theta-\psi)\operatorname{div} \mathbf{b}) \\
- 4\tau^2(\xi^2+\kappa^2) - 12\tau^2\kappa(\kappa + \operatorname{div} \mathbf{n})\Big]\,\frac{\delta\kappa}{\delta \mathbf{b}} - 3(\xi^2-\kappa^2)\tau \operatorname{div} \mathbf{b}\,\frac{\delta}{\delta \mathbf{b}}\,(\theta-\psi)
\end{aligned}
$$

$$+ [2\xi^2\kappa \operatorname{div} \underset{\sim}{b} - 3\tau(\theta-\psi)(\xi^2 - \kappa^2)] \frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b}$$

$$- 6[3\xi^2 + \kappa^2]\tau^2 \frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) = 0 ,$$

and on substituting for $\frac{\delta\kappa}{\delta b}$, $\frac{\delta}{\delta b}(\theta-\psi)$, $\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b}$ and $\frac{\delta\xi^2}{\delta b}$ from (2.11), (2.25), (2.29) and (2.30) we get

$$\begin{aligned} &- 6[3\xi^2 + \kappa^2]\tau^2 \frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) + 12\tau^2(\kappa^2 - 6\xi^2)(\kappa + \operatorname{div} \underset{\sim}{n})\operatorname{div} \underset{\sim}{b} \\ &\quad - 144\kappa(\theta-\psi)\tau^3(\kappa + \operatorname{div} \underset{\sim}{n}) + \xi^2\tau(\operatorname{div} \underset{\sim}{b})^3 + 5(-3\xi^2 + \kappa^2)(\theta-\psi)\tau(\operatorname{div} \underset{\sim}{b})^2 \\ &\quad - 24[(\theta-\psi)^2 + \xi^2]\tau^2\kappa \operatorname{div} \underset{\sim}{b} + 8(3\xi^2 - 7\kappa^2)\tau^3(\theta-\psi) \\ &\quad + \frac{3(\xi^2-\kappa^2)}{\kappa} (\theta-\psi)\tau^2[6(\theta-\psi)\operatorname{div} \underset{\sim}{b} + 36\tau(\kappa + \operatorname{div} \underset{\sim}{n})] = 0 . \end{aligned} \tag{4.15}$$

We note that the integral (4.14) can be written

$$\begin{aligned} &\tau(3\xi^2 + \kappa^2)[(\theta-\psi)\operatorname{div} \underset{\sim}{b} + 6\tau(\kappa + \operatorname{div} \underset{\sim}{n})] \\ &\qquad = \xi^2\kappa(\operatorname{div} \underset{\sim}{b})^2 - 4\tau^2\kappa(\xi^2+\kappa^2) + 4\kappa^2\tau(\theta-\psi)\operatorname{div} \underset{\sim}{b} . \end{aligned} \tag{4.16}$$

We write the last term on the left hand side of (4.15) in the form

$$\begin{aligned} &\frac{6(3\xi^2 + \kappa^2)}{\kappa} [(\theta-\psi)\operatorname{div} \underset{\sim}{b} + 6\tau(\kappa + \operatorname{div} \underset{\sim}{n})](\theta-\psi)\tau^2 \\ &- 24\kappa(\theta-\psi)\tau^2[(\theta-\psi)\operatorname{div} \underset{\sim}{b} + 6\tau(\kappa + \operatorname{div} \underset{\sim}{n})] \end{aligned}$$

and substitute for the first term the expression (4.16). Since κ cannot vanish it may be cancelled. We obtain the formula

$$- 6[3\xi^2 + \kappa^2]\tau^2 \frac{\delta}{\delta b} (\kappa + \operatorname{div} \mathbf{n}) + 12\tau^2(\kappa^2 - 6\xi^2)(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b}$$

$$- 288\kappa(\theta-\psi)\tau^3(\kappa + \operatorname{div} \mathbf{n}) - 24\tau^2\kappa[(\theta-\psi)^2 + \xi^2] \operatorname{div} \mathbf{b}$$

$$+ (-9\xi^2 + 5\kappa^2)(\theta-\psi)\tau(\operatorname{div} \mathbf{b})^2 + \xi^2\kappa(\operatorname{div} \mathbf{b})^3 - 32\tau^3\kappa^2(\theta-\psi) = 0 \ . \qquad (4.17)$$

According to (2.26) the term $\frac{\delta}{\delta b} (\kappa + \operatorname{div} \mathbf{n})$ may be replaced by $\frac{\delta}{\delta n} \operatorname{div} \mathbf{b}$ in (4.17) and all other relations where it occurs.

We must determine now the condition representing the directional derivative of (2.18) with respect to $\mathbf{b}$. Using (2.10), (2.11) and (2.12) we get

$$\frac{\delta^2}{\delta b \delta s} \operatorname{div} \mathbf{b} + 4\tau\kappa \operatorname{div} \mathbf{b} + \psi \frac{\delta}{\delta b} \operatorname{div} \mathbf{b}$$

$$+ \operatorname{div} \mathbf{b} \, [- (\psi-\theta)\operatorname{div} \mathbf{b} + 6\tau(\kappa + \operatorname{div} \mathbf{n})]$$

$$- 2\tau \frac{\delta}{\delta b} (\kappa + \operatorname{div} \mathbf{n}) = 0 \ . \qquad (4.18)$$

Also by (2.8) and (2.29)

$$\frac{\delta^2}{\delta b \delta s} \operatorname{div} \mathbf{b} = \frac{\delta^2}{\delta s \delta b} \operatorname{div} \mathbf{b} + 4\tau \frac{\delta}{\delta n} \operatorname{div} \mathbf{b} + \theta \frac{\delta}{\delta b} \operatorname{div} \mathbf{b}$$

$$= \frac{\delta}{\delta s} \left[(\operatorname{div} \mathbf{b})^2 - 8\tau^2 - \frac{6\tau}{\kappa} (\theta-\psi)\operatorname{div} \mathbf{b} - \frac{36\tau^2}{\kappa} (\kappa + \operatorname{div} \mathbf{n})\right]$$

$$+ 4\tau \frac{\delta}{\delta n} \operatorname{div} \mathbf{b} + \theta \frac{\delta}{\delta b} \operatorname{div} \mathbf{b} \ . \qquad (4.19)$$

Expanding (4.19) using the formulae (2.14), (2.15), (2.18), (2.26) and (2.29) and eliminating $\frac{\delta^2}{\delta b \delta s}$ div b from (4.18) and reducing we get*

$$\frac{1}{\kappa}\left[6\tau(\theta-\psi)\operatorname{div}\underset{\sim}{b} + 36\tau^2(\kappa + \operatorname{div}\underset{\sim}{n})\right]\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) - 6\tau\operatorname{div}\underset{\sim}{b}\,\frac{\delta}{\delta s}(\theta-\psi)$$

$$+ 2\tau\kappa\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} - \left(16\tau\kappa + 6(\theta-\psi)\operatorname{div}\underset{\sim}{b} + 72\tau(\kappa + \operatorname{div}\underset{\sim}{n})\right)\frac{\delta\tau}{\delta s}$$

$$+ 10\tau\kappa(\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b} + 2(\theta-\psi)\kappa(\operatorname{div}\underset{\sim}{b})^2$$

$$+ 12\tau\left[\theta(\psi-\theta) + \kappa^2 - 12\tau^2\right]\operatorname{div}\underset{\sim}{b} - 24\tau^2(2\theta + \psi)(\kappa + \operatorname{div}\underset{\sim}{n})$$

$$+ 4\tau^2\kappa(\theta + 4\psi) = 0 \quad . \tag{4.20}$$

---

*We note from (2.8), (2.27) and (2.28) that $\frac{\delta}{\delta b}\left(\frac{\delta^2\tau}{\delta s^2}\right) = 4\tau\kappa\frac{\delta\tau}{\delta s}$.

If one evaluates this expression using (2.14) and the appropriate formulae for the gradients, one obtains the condition (4.20). This gives a check on (4.20).

5. LEMMA I

We now prove

LEMMA I. There exist a functional relation $f\left(\frac{\kappa}{\tau}, \frac{\theta-\psi}{\tau}\right) = 0.$

We generate four polynomial conditions involving the variables $\frac{\delta\kappa}{\delta n}$, $\frac{\delta}{\delta n}(\theta-\psi)$, div $\underset{\sim}{b}$, $\theta-\psi$, κ and τ. We show that we can eliminate the three variables $\frac{\delta\kappa}{\delta n}$, $\frac{\delta}{\delta n}(\theta-\psi)$ and div $\underset{\sim}{b}$ from these conditions. We show that there exist a homogeneous polynomial relation in the three variables $\theta-\psi$, κ and τ. Since τ cannot vanish, the functional relation $f\left(\frac{\kappa}{\tau}, \frac{\theta-\psi}{\tau}\right) = 0$ follows by division by τ to a power equal to the total degree of the polynomial.

We make the numerical substitutions

$$\theta-\psi = \tau = \kappa = 1, \quad \ell^2 = (\theta-\psi)^2 + 36\tau^2 = 37 \,. \tag{5.1}$$

At no stage in the elimination is the degree of the expressions in $\frac{\delta\kappa}{\delta n}$, $\frac{\delta}{\delta n}(\theta-\psi)$, or div $\underset{\sim}{b}$ affected by these substitutions. Complicated functions of the variables $\theta-\psi$, τ, κ will be represented as numbers. We arrive at two algebraic equations in div $\underset{\sim}{b}$, with numerical coefficients. We show that the eliminant of div $\underset{\sim}{b}$ from these two equations does not vanish identically, by showing that it does not vanish for the above values. We do this by verifying that the two equations do not have a common root. The required condition then follows from the requirement that the eliminant must necessarily be zero.

For the particular values (5.1), the conditions (2.29) and (2.31) reduce respectively to

$$\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} = (\operatorname{div} \underset{\sim}{b})^2 - 6 \operatorname{div} \underset{\sim}{b} - 36(\kappa + \operatorname{div} \underset{\sim}{n}) - 8 \ , \qquad (5.2)$$

and

$$\frac{\delta}{\delta s} (\psi - 0) = \frac{\delta\kappa}{\delta n} + 12 + 2\psi - (\kappa + \operatorname{div} \underset{\sim}{n}) \ . \qquad (5.3)$$

The condition (4.5) becomes

$$\frac{\delta\kappa}{\delta s} + 6\kappa = \frac{\operatorname{div} \underset{\sim}{b}}{2} \frac{\delta\kappa}{\delta n} + \frac{1}{2} \frac{\delta}{\delta n} \operatorname{div} b + \frac{\delta}{\delta n} (\psi - 0) - \frac{2}{3} \operatorname{div} \underset{\sim}{b}$$

$$- 3(\kappa + \operatorname{div} n) - \frac{4}{3} \ . \qquad (5.4)$$

The conditions (4.5) and (4.8) become respectively

$$60\psi = 4 \frac{\delta\kappa}{\delta n} - \operatorname{div} \underset{\sim}{b} \frac{\delta}{\delta n} (\psi - 0) + 2\left[\frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} + (\kappa + \operatorname{div} \underset{\sim}{n}) \operatorname{div} \underset{\sim}{b}\right]$$

$$+ 440(\kappa + \operatorname{div} n) - 24(\operatorname{div} b)^2 + 72 \operatorname{div} \underset{\sim}{b} + 150 \ , \qquad (5.5)$$

and

$$\psi = \frac{1}{3}\left[\left(\frac{\delta\kappa}{\delta s} + 6\kappa\right) + \frac{\delta}{\delta n} (\psi - 0)\right] + \frac{109}{18} \operatorname{div} \underset{\sim}{b} + \frac{1}{9} \ . \qquad (5.6)$$

The basic integral (4.14) takes the form

$$56(12)(\kappa + \operatorname{div} \underset{\sim}{n}) = 37(\operatorname{div} \underset{\sim}{b})^2 - 108 \operatorname{div} \underset{\sim}{b} - 144 \ , \qquad (5.7)$$

and by squaring this we have

$$56(12)56(12)(\kappa + \operatorname{div} \underset{\sim}{n})^2 = 1369(\operatorname{div} \underset{\sim}{b})^4 - 7992(\operatorname{div} \underset{\sim}{b})^3 + 1008(\operatorname{div} \underset{\sim}{b})^2$$

$$+ 31104 \operatorname{div} \underset{\sim}{b} + 20736 \ . \tag{5.8}$$

From (4.17) we have

$$6(112) \frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) = -12(221)(\kappa + \operatorname{div} \underset{\sim}{n})\operatorname{div} \underset{\sim}{b} - 288(\kappa + \operatorname{div} \underset{\sim}{n})$$

$$+ 37(\operatorname{div} \underset{\sim}{b})^3 - 328(\operatorname{div} \underset{\sim}{b})^2$$

$$- 24(38)\operatorname{div} \underset{\sim}{b} - 32 \ . \tag{5.9}$$

On multiplying (5.9) by 56 and eliminating $(\kappa + \operatorname{div} \underset{\sim}{n})$ by (5.7) we get

$$56(56)(12) \frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) = -6105(\operatorname{div} \underset{\sim}{b})^3 + 4612(\operatorname{div} \underset{\sim}{b})^2$$

$$- 16656 \operatorname{div} \underset{\sim}{b} + 1664 \ . \tag{5.10}$$

We proceed to derive the four basic relations required to prove Lemma 1.

Relation I

Substituting the values (5.1) in the expression (4.20), and noting from (2.14) that

$$\frac{\delta\psi}{\delta s} = \frac{\operatorname{div} \underset{\sim}{b}}{3} [illegible] \frac{2}{3} [illegible] 2\psi \ . \tag{5.11}$$

We get, on using (2.26)

$$[6 \text{ div } \underset{\sim}{b} + 36(\kappa + \text{div } \underset{\sim}{n})]\left(\frac{\delta\kappa}{\delta s} + \delta\kappa\right) - 6 \text{ div } \underset{\sim}{b}\, \frac{\delta}{\delta s}(\theta - \psi) + 2\frac{\delta}{\delta b}(\kappa + \text{div } \underset{\sim}{b})$$

$$+ \left[16 + 6 \text{ div } \underset{\sim}{b} + 72(\kappa + \text{div } \underset{\sim}{n})\right]\left(\frac{\text{div } \underset{\sim}{b}}{3} + \frac{2}{3} + 2\psi\right) + 2(\text{div } \underset{\sim}{b})^2$$

$$- 12(\theta + 11)\text{div } \underset{\sim}{b} + 10(\kappa + \text{div } \underset{\sim}{n})\text{div } \underset{\sim}{b} - 24(2\theta + \psi)(\kappa + \text{div } \underset{\sim}{n})$$

$$+ 4(\theta + 4\psi) = 0 . \tag{5.12}$$

Multiplying (4.1) by 6 and making the substitution (5.1) we get

$$6 \text{ div } \underset{\sim}{b}\, \frac{\delta}{\delta s}(\psi - \theta) + 24\frac{\delta}{\delta n}(\psi - \theta) + 6\left[\frac{\delta}{\delta b}(\kappa + \text{div } \underset{\sim}{n}) + (\kappa + \text{div } \underset{\sim}{n})\text{div } \underset{\sim}{b}\right]$$

$$+ \text{div } \underset{\sim}{b}\, [-6(\psi + \theta) + 144] - 12(\theta + 2\psi) - 36(\kappa + \text{div } \underset{\sim}{n}) = 0 \tag{5.13}$$

Subtracting (5.12) and (5.13) to eliminate $\frac{\delta}{\delta s}(\psi - \theta)$ we get

$$[6 \text{ div } \underset{\sim}{b} + 36(\kappa + \text{div } \underset{\sim}{n})]\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) - 24\frac{\delta}{\delta n}(\psi - \theta) - 4\frac{\delta}{\delta b}(\kappa + \text{div } \underset{\sim}{n})$$

$$+ 28(\kappa + \text{div } \underset{\sim}{n})\text{div } \underset{\sim}{b} + 4(\text{div } \underset{\sim}{b})^2 - \frac{818}{3}\text{div } \underset{\sim}{b} + 36(\kappa + \text{div } \underset{\sim}{n})$$

$$+ \frac{80}{3} + 3\psi\left(4 \text{ div } \underset{\sim}{b} + 24(\kappa + \text{div } \underset{\sim}{n}) + \frac{88}{3}\right) = 0 . \tag{5.14}$$

On substituting for ψ from (5.6) we get

$$- 4 \frac{\delta}{\delta b} (\kappa + \operatorname{div} \underset{\sim}{n}) + \left[10 \operatorname{div} \underset{\sim}{b} + 60(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{88}{3}\right] \left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right)$$

$$+ \left[4 \operatorname{div} \underset{\sim}{b} + 24(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{16}{3}\right] \frac{\delta}{\delta n} (\psi - \theta) + 464(\kappa + \operatorname{div} \underset{\sim}{n}) \operatorname{div} \underset{\sim}{b}$$

$$+ \frac{230}{3} (\operatorname{div} \underset{\sim}{b})^2 + \frac{2354}{9} \operatorname{div} \underset{\sim}{b} + 44(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{328}{9} = 0 \; . \qquad (5.15)$$

Substituting for $\frac{\delta\kappa}{\delta s} + \theta\kappa$ from (5.4) and using (2.26) we obtain

$$\operatorname{div} \underset{\sim}{b} \left(5 \operatorname{div} \underset{\sim}{b} + 30(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{44}{3}\right) \frac{\delta\kappa}{\delta n}$$

$$+ \left(5 \operatorname{div} \underset{\sim}{b} + 30(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{32}{3}\right) \frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b}$$

$$+ \left(14 \operatorname{div} \underset{\sim}{b} + 84(\kappa + \operatorname{div} \underset{\sim}{n}) + \frac{104}{3}\right) \frac{\delta}{\delta n} (\psi - \theta) + 394(\kappa + \operatorname{div} \underset{\sim}{n}) \operatorname{div} \underset{\sim}{b}$$

$$- 180(\kappa + \operatorname{div} \underset{\sim}{n})^2 - 124(\kappa + \operatorname{div} \underset{\sim}{n}) + 70(\operatorname{div} \underset{\sim}{b})^2 + \frac{686}{3} \operatorname{div} \underset{\sim}{b}$$

$$- \frac{8}{3} = 0 \quad . \qquad (5.16)$$

Finally eliminating $(\kappa + \operatorname{div} \underset{\sim}{n})$ and $\frac{\delta}{\delta n} \operatorname{div} \underset{\sim}{b} = \frac{\delta}{\delta n} (\kappa + \operatorname{div} \underset{\sim}{n})$ in favour of $\operatorname{div} \underset{\sim}{b}$ using (5.7), (5.8) and (5.10) we obtain*

*In presenting this development we have made the numerical substitutions directly from the beginning. Originally however, the condition (5.16) was obtained in general form with the variables θ-ψ, τ, κ and ξ^2 maintained. The numerical values were then substituted in this expression. This gives a check on (5.16). The reduction to the form (5.17) was checked by setting div $\underset{\sim}{b}$ = 1 before and after collecting terms and ensuring that the same sums were obtained.

$$(56)^2\ [2220\ (\text{div}\ \underset{\sim}{b})^3 + 240(\text{div}\ \underset{\sim}{b})^2 + 11072\ \text{div}\ \underset{\sim}{b}]\ \frac{\delta\kappa}{\delta n}$$

$$+\ (56)^2\ [6216(\text{div}\ \underset{\sim}{b})^2 + 672\ \text{div}\ \underset{\sim}{b} + 22400]\ \frac{\delta}{\delta n}\ (\psi-\theta)$$

$$-\ 1129425(\text{div}\ \underset{\sim}{b})^5 - 1568800(\text{div}\ \underset{\sim}{b})^4 + 98972816(\text{div}\ \underset{\sim}{b})^3$$

$$-\ \frac{470272}{3}\ (\text{div}\ \underset{\sim}{b})^2 + 651799552\ \text{div}\ \underset{\sim}{b} + \frac{200120320}{3} = 0\ . \tag{5.17}$$

Relation 2

We propose now to take the gradient of (4.5) with respect to $\underset{\sim}{b}$. Noting the conditon (2.26), we will obtain an expression for $\frac{\delta^2}{\delta b^2}(\kappa + \text{div}\ \underset{\sim}{n})$. A second expression for $\frac{\delta^2}{\delta b^2}(\kappa + \text{div}\ \underset{\sim}{n})$ will follow from the $\underset{\sim}{b}$-gradient of (4.17). From these two expressions we shall eventually obtain a second relation of the general form of (5.17).

To facilitate the algebra we write

$$\eta = \frac{\delta}{\delta n}\ (\psi-\theta) - \left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right). \tag{5.18}$$

Then by (2.33), (4.9) and (4.10)

$$\frac{\delta\eta}{\delta b} = -\ \text{div}\ \underset{\sim}{b}\eta - 8\mu\kappa^2 \tag{5.19}$$

where μ, given by (2.33), is constant along a $\underset{\sim}{b}$-line.

Using (2.26), (2.33) and (5.18) we write (4.5) in the form

$$2\ \text{div}\ \underset{\sim}{b}\left[\mu\kappa^2 - \kappa(\kappa + \text{div}\ \underset{\sim}{n})\right] + 2\kappa\frac{\delta}{\delta b}(\kappa + \text{div}\ \underset{\sim}{n}) + 4\tau\eta$$

$$-2\left(\frac{\kappa^2}{3} + (\theta-\psi)^2\right) \operatorname{div} b - 12\tau(\theta-\psi)(\kappa + \operatorname{div} n)$$

$$-\frac{16}{3}(\theta-\psi)\tau\kappa = 0 . \tag{5.20}$$

Using (2.11), (2.25) and (5.19) we get

$$2\frac{\delta}{\delta b}\operatorname{div} b\left(\mu\kappa^2 - \kappa(\kappa + \operatorname{div} n)\right)$$

$$+ 2\operatorname{div} b\left(-2\mu\kappa^2 \operatorname{div} b - \kappa\frac{\delta}{\delta b}(\kappa + \operatorname{div} n) + \kappa \operatorname{div} b(\kappa + \operatorname{div} n)\right)$$

$$- 2\kappa \operatorname{div} b\frac{\delta}{\delta b}(\kappa + \operatorname{div} n) + 2\kappa\frac{\delta^2}{\delta b^2}(\kappa + \operatorname{div} n) + 4\tau\left(-\operatorname{div} b\eta - 8\tau\mu\kappa^2\right)$$

$$- 2\left(-\frac{2\kappa^2}{3}\operatorname{div} b + 8\tau\kappa(\theta-\psi)\right)\operatorname{div} b - 2\left(\frac{\kappa^2}{3} + (\theta-\psi)^2\right)\frac{\delta}{\delta b}\operatorname{div} b$$

$$- 48\tau^2\kappa(\kappa + \operatorname{div} n) - 12\tau(\theta-\psi)\frac{\delta}{\delta b}(\kappa + \operatorname{div} n)$$

$$- \frac{64}{3}\tau^2\kappa^2 + \frac{16}{3}(\theta-\psi)\tau\kappa \operatorname{div} b = 0 .$$

Substituting the expression (5.2) for $\frac{\delta}{\delta b}$ div b, replacing μ by the expression (2.33) and replacing η by the expression (5.18) we get

$$2\kappa\frac{\delta^2}{\delta b^2}(\kappa + \operatorname{div} n) - 12\left[4\tau^2 + (\theta-\psi)\operatorname{div} b + 6\tau(\kappa + \operatorname{div} n)\right]\frac{\delta\kappa}{\delta n}$$

$$- 2\left[\kappa \operatorname{div} b + 6\tau(\theta-\psi)\right]\frac{\delta}{\delta b}(\kappa + \operatorname{div} n) - 2(\kappa + \operatorname{div} n)(\operatorname{div} b)^2$$

$$- 12\tau(\theta-\psi)(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b} + \left[\frac{72\tau^2}{\kappa} (\theta-\psi)^2 - 56\tau^2\kappa \right](\kappa + \operatorname{div} \mathbf{n})$$

$$- 4(\theta-\psi)^2(\operatorname{div} \mathbf{b})^2 + \left(-12\kappa\tau + \frac{12\tau}{\kappa} (\theta-\psi)^2\right) (\theta-\psi)\operatorname{div} \mathbf{b}$$

$$+ 16 \left[(\theta-\psi)^2 - \kappa^2\right] = 0 \ .$$

Substituting the particular values (5.1) we then obtain*

$$2 \frac{\delta^2}{\delta b^2} (\kappa + \operatorname{div} \mathbf{n}) - 2(6 \operatorname{div} \mathbf{b} + 36(\kappa + \operatorname{div} \mathbf{n}) + 24) \frac{\delta\kappa}{\delta n}$$

$$- 2(\operatorname{div} \mathbf{b} + 6) \frac{\delta}{\delta b} (\kappa + \operatorname{div} \mathbf{n}) - 2(\kappa + \operatorname{div} \mathbf{n})(\operatorname{div} \mathbf{b})^2$$

$$- 12(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b} + 16(\kappa + \operatorname{div} \mathbf{n}) - 4(\operatorname{div} \mathbf{b})^2 = 0 \ . \tag{5.21}$$

Taking the directional derivative with respect to $\mathbf{b}$ of (4.17), we have

---

*We note that in the specialization (5.21), the coefficient of div $\mathbf{b}$ and the term independnet of $(\kappa + \operatorname{div} \mathbf{n})$ and div $\mathbf{b}$ go out. Since the final required expression (5.24) is obtained by eliminating $\frac{\delta^2}{\delta b^2} (\kappa + \operatorname{div} \mathbf{n})$ between (5.21) and (5.22), terms linear in div $\mathbf{b}$ and independent of div $\mathbf{b}$ are maintained in the final result. The order of the final expressions in div $\mathbf{b}$ will not be affected.

$$-6\,[3\xi^2 + \kappa^2]\tau^2 \frac{\delta^2}{\delta b^2}(\kappa + \operatorname{div} \mathbf{n}) + \Big[-18\tau^2 \frac{\delta}{\delta b}(\kappa + \operatorname{div} \mathbf{n})$$

$$- 72\tau^2(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b} - 24\tau^2 \operatorname{div} \mathbf{b} - 9(\theta-\psi)\tau(\operatorname{div} \mathbf{b})^2$$

$$+ \kappa(\operatorname{div} \mathbf{b})^3\Big] \frac{\delta\xi}{\delta b}^2$$

$$+ \Big[-12\kappa\tau^2 \frac{\delta}{\delta b}(\kappa + \operatorname{div} \mathbf{n}) + 24\kappa\tau^3(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b}$$

$$- 288(\theta-\psi)\tau^3(\kappa + \operatorname{div} \mathbf{n}) - 24\tau^2[(\theta-\psi)^2 + \xi^2]\operatorname{div} \mathbf{b}$$

$$+ 10\kappa(\theta-\psi)\tau(\operatorname{div} \mathbf{b})^2 + \xi^2(\operatorname{div} \mathbf{b})^3 - 64\tau^3\kappa(\theta-\psi)\Big] \frac{\delta\kappa}{\delta b}$$

$$+ [-288\kappa\tau^3(\kappa + \operatorname{div} \mathbf{n}) - 48\tau^2\kappa(\theta-\psi)\operatorname{div} \mathbf{b}$$

$$+ (-9\xi^2 + 5\kappa^2)\tau(\operatorname{div} \mathbf{b})^2 - 32\tau^3\xi^2] \frac{\delta}{\delta b}(\theta-\psi)$$

$$+ [12\tau^2(\kappa^2 - 6\xi^2)\operatorname{div} \mathbf{b} - 288\kappa(\theta-\psi)\tau^3] \frac{\delta}{\delta b}(\kappa + \operatorname{div} \mathbf{n})$$

$$+ \Big[12\tau^2(\kappa^2 - 6\xi^2)(\kappa + \operatorname{div} \mathbf{n}) - 24\tau^3[(\theta-\psi)^2 + \xi^2]$$

$$+ 2(-9\xi^2 + 5\kappa^2)(\theta-\psi)\tau \operatorname{div} \mathbf{b} + 3\kappa\tau(\operatorname{div} \mathbf{b})^2\Big] \frac{\delta}{\delta b}(\operatorname{div} \mathbf{b}) = 0 .$$

Substituting the expressions (2.11), (2.25), (2.29) and (2.39) for $\frac{\delta\kappa}{\delta b}$, $\frac{\delta}{\delta b}(\theta-\psi)$, $\frac{\delta}{\delta b}\operatorname{div} \mathbf{b}$ and $\frac{\delta\xi}{\delta b}^2$ and substituting the particular values (5.1) reducing and dividing by 2 we get*

$$336 \frac{\delta^2}{\delta b^2} (\kappa + \operatorname{div} \mathbf{n}) = - (216 + 1320 \operatorname{div} \mathbf{b}) \frac{\delta}{\delta b} (\kappa + \operatorname{div} \mathbf{n})$$

$$- 3356(\kappa + \operatorname{div} \mathbf{n})(\operatorname{div} \mathbf{b})^2 + 47736(\kappa + \operatorname{div} \mathbf{n})^2$$

$$+ 19620(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b} + 26448(\kappa + \operatorname{div} \mathbf{n}) + 37(\operatorname{div} \mathbf{b})^4$$

$$- 662(\operatorname{div} \mathbf{b})^3 + 832(\operatorname{div} \mathbf{b})^2 + 5200 \operatorname{div} \mathbf{b} + 3584 \, . \tag{5.22}$$

Multiplying (5.21) by 168 and eliminating $\frac{\delta^2}{\delta b^2} (\kappa + \operatorname{div} \mathbf{n})$ between (5.21) and (5.22) we then obtain

$$- 336 \, [6 \operatorname{div} \mathbf{b} + 36(\kappa + \operatorname{div} \mathbf{n}) + 21] \frac{\delta \kappa}{\delta n}$$

$$- (1656 \operatorname{div} \mathbf{b} + 2232) \frac{\delta}{\delta b} (\kappa + \operatorname{div} \mathbf{n}) - 5672(\kappa + \operatorname{div} \mathbf{n})(\operatorname{div} \mathbf{b})^2$$

$$+ 47736(\kappa + \operatorname{div} \mathbf{n})^2 + 17604(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b} + 29136(\kappa + \operatorname{div} \mathbf{n})$$

$$+ 37(\operatorname{div} \mathbf{b})^4 - 662(\operatorname{div} \mathbf{b})^3 + 160(\operatorname{div} \mathbf{b})^2 + 5200 \operatorname{div} \mathbf{b}$$

$$+ 3584 = 0 \, . \tag{5.23}$$

---

*The condition (5.21) and (5.22) were obtained in two ways. First by making the numerical substitution for $\partial\psi$, t, κ and δ^2 at the outset and collecting terms, and then by collecting terms to get the general forms of (5.21) and (5.22) before substituting numerical values for the variables.

We also note that the cancelled factor 2 in the derivation of (5.22) is a numerical factor in the general case.

Finally, substituting the expression (5.7), (5.8) and (5.10) for $(\kappa + \text{div}\ \underset{\sim}{n})^2$ and $\frac{\delta}{\delta b}(\kappa + \text{div}\ \underset{\sim}{n})$ into (5.23) and reducing we obtain*

$$7(56)(48)\ [-666(\text{div}\ \underset{\sim}{b})^2 - 72\ \text{div}\ \underset{\sim}{b} - 5472]\ \frac{\delta\kappa}{\delta n} + 466988(\text{div}\ \underset{\sim}{b})^4$$

$$+ 3984036(\text{div}\ \underset{\sim}{b})^3 + 5415552(\text{div}\ \underset{\sim}{b})^2 + 17832768\ \text{div}\ \underset{\sim}{b}$$

$$- 10652928 = 0\ . \qquad (5.24)$$

Relation 3

This relation is obtained by taking the directional derivative of (4.14) with respect to $\underset{\sim}{n}$. From (2.10) and (2.38) we see that

$$\frac{\delta \epsilon^2}{\delta n} = 2(\theta-\psi)\ \frac{\delta}{\delta n}\ (\theta-\psi)\ . \qquad (5.25)$$

Using (5.25) and also the expression (2.30) for $\frac{\delta}{\delta n}(\kappa + \text{div}\ \underset{\sim}{n})$ we have

$$\Big[\kappa(\text{div}\ \underset{\sim}{b})^2 - \text{[illegible]}[\text{[illegible]} + 3(\theta-\psi)\text{div}\ \underset{\sim}{b}] - 18\text{[illegible]}^2(\kappa + \text{div}\ \underset{\sim}{n})\Big]\, 2(\theta-\psi)\ \frac{\delta}{\delta n}\ (\theta-\psi)$$

$$- 3\text{[illegible]}(\text{[illegible]}^2 - \text{[illegible]}^2)\ \text{div}\ \underset{\sim}{b}\ \frac{\delta}{\delta n}\ (\theta-\psi) + \Big[\text{[illegible]}^2(\text{div}\ \underset{\sim}{b})^2$$

$$+ 2\text{[illegible]}[\text{[illegible]} + 3(\theta-\psi)\text{div}\ \underset{\sim}{b}] - 4\text{[illegible]}^2(\text{[illegible]}^2 - \text{[illegible]}^2) - 12\text{[illegible]}^2(\kappa + \text{div}\ \underset{\sim}{n})\Big]\ \frac{\delta\kappa}{\delta n}$$

$$+ [2\text{[illegible]}^2\ \text{div}\ \underset{\sim}{b} - 3\text{[illegible]}(\text{[illegible]}^2 - \text{[illegible]}^2)\text{[illegible]}]\ \text{[illegible]}_n\ \text{div}\ \underset{\sim}{b}$$

*Here again the arithmetrical reduction was checked by comparing values for div b = 1 before and after collecting terms of the various powers of div b.

$$- 6[3\kappa^2 + \tau^2]\,\tau^2 \Big[-2(\mathrm{div}\,\mathbf{b})^2 - (\kappa + \mathrm{div}\,\mathbf{n})^2 - [\theta\psi - 17\tau^2]$$

$$+ \frac{2\tau}{\kappa}\,[3(\theta-\psi)\mathrm{div}\,\mathbf{b} + 18\tau(\kappa + \mathrm{div}\,\mathbf{n})]\Big] = 0 \,. \qquad (5.26)$$

Using (5.5) to eliminate $\delta\psi$ from (5.26), and substituting the special values (5.1), we obtain on reduction

$$\left(-2(\mathrm{div}\,\mathbf{b})^2 + 2\,\mathrm{div}\,\mathbf{b} + 36(\kappa + \mathrm{div}\,\mathbf{n}) + 8\right)\frac{\delta}{\delta n}\,(\psi-\theta)$$

$$+ \left(37(\mathrm{div}\,\mathbf{b})^2 + 6\,\mathrm{div}\,\mathbf{b} - 12(\kappa + \mathrm{div}\,\mathbf{n}) + 312\right)\frac{\delta\kappa}{\delta n}$$

$$+ (74\,\mathrm{div}\,\mathbf{b} + 116)\,\frac{\delta}{\delta n}\,\mathrm{div}\,\mathbf{b} + 224\,(\kappa + \mathrm{div}\,\mathbf{n})\mathrm{div}\,\mathbf{b}$$

$$- 1344(\mathrm{div}\,\mathbf{b})^2 + 672(\kappa + \mathrm{div}\,\mathbf{n})^2 + 25088(\kappa + \mathrm{div}\,\mathbf{n})$$

$$+ 4032\,\mathrm{div}\,\mathbf{b} + 5376 = 0 \,. \qquad (5.27)$$

Substituting the expressions (5.7), (5.8) and (5.10) for $\kappa + \mathrm{div}\,\mathbf{n}$, $(\kappa + \mathrm{div}\,\mathbf{n})^2$ and $\frac{\delta}{\delta b}(\kappa + \mathrm{div}\,\mathbf{n}) = \frac{\delta}{\delta n}\,\mathrm{div}\,\mathbf{b}$ into (5.27) we get*

---

*As a check on (5.27) one obtained this condition first by specializing (5.26) and using (5.5) and collecting terms, and second by obtaining the general form of (5.27) by substituting the expression (4.3) for $\theta\psi$ into (5.26) reducing, and substituting the numerical values in the final form.

The numerical reduction to (5.28) was checked by substituting $\mathrm{div}\,\mathbf{b} = 1$, before and after collecting terms.

$$336\,[-(\text{div}\,\underset{\sim}{b})^2 - 212\,\text{div}\,\underset{\sim}{b} + 16]\,\frac{\delta}{\delta n}\,(\psi-\theta)$$

$$+\ 336\,[2035\,(\text{div}\,\underset{\sim}{b})^2 + 444\,\text{div}\,\underset{\sim}{b} + 17616]\,\frac{\delta\tau}{\delta n}$$

$$-\ 187553\,(\text{div}\,\underset{\sim}{b})^4 - 175158\,(\text{div}\,\underset{\sim}{b})^3 - 295464\,(\text{div}\,\underset{\sim}{b})^2$$

$$-\ 936736\,\text{div}\,\underset{\sim}{b} + 677120 = 0\ . \tag{5.28}$$

Relation 4

This condition is derived from the directional derivative of (4.14) with respect to $\underset{\sim}{s}$. We note from (2.38) that

$$\frac{\delta\varepsilon^2}{\delta s} = 2(\psi-\theta)\,\frac{\delta}{\delta s}\,(\psi-\theta) + 72\tau\,\frac{\delta\tau}{\delta s}\ . \tag{5.29}$$

Using (5.29) and the expressions (2.15) and (2.18) for $\frac{\delta}{\delta s}\,(\kappa + \text{div}\,\underset{\sim}{n})$ and $\frac{\delta}{\delta s}\,\text{div}\,b$, we obtain

$$\Big[\kappa\,(\text{div}\,\underset{\sim}{b})^2 - \tau\,[4\tau\kappa + 3(\theta-\psi)\text{div}\,\underset{\sim}{b}] - 18\tau^2(\kappa + \text{div}\,\underset{\sim}{n})\Big]$$

$$\times \Big[2(\psi-\theta)\,\frac{\delta}{\delta s}\,(\psi-\theta) + 72\tau\,\frac{\delta\tau}{\delta s}\Big] - 3\tau(\varepsilon^2-\kappa^2)\text{div}\,\underset{\sim}{b}\,\frac{\delta}{\delta s}\,(\theta-\psi)$$

$$+ \Big[-8\tau\kappa\,(\varepsilon^2-\kappa^2) - 3(\theta-\psi)\text{div}\,b(\varepsilon^2-\kappa^2) - 12(3\varepsilon^2 + \kappa^2)\tau\,(\kappa + \text{div}\,\underset{\sim}{n})\Big]\,\frac{\delta\tau}{\delta s}$$

$$+ \Big[\varepsilon^2(\text{div}\,b)^2 + 2\tau\,[4\tau\kappa + 3(\theta-\psi)\text{div}\,b] - 4\tau^2(\varepsilon^2-\kappa^2)$$

$$- 12\kappa\tau^2(\kappa + \text{div}\,\underset{\sim}{b})\Big]\,\frac{\delta\kappa}{\delta s}$$

$$+ [2\varepsilon^2\,\text{div}\,b - 3(\varepsilon^2-\kappa^2)\tau(\theta-\psi)][4\tau\kappa + 3(\theta-\psi)\text{div}\,\underset{\sim}{b} + 2\tau(\kappa + \text{div}\,\underset{\sim}{n})]$$

$$- 6[3\kappa^2 + \iota^2]\,\tau^2\,[4\iota \operatorname{div} \mathbf{b} - (\theta-\psi)(2\kappa + \operatorname{div} \mathbf{n})$$

$$- \psi(2\kappa + \operatorname{div} \mathbf{n})] = 0 \,. \qquad (5.30)$$

On substituting the values (5.1), and thus writing, for $\theta-\psi = 1$, $\frac{\delta\kappa}{\delta s} = \left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) - (1 + \psi)\kappa$, we reduce (5.30) to

$$[-111(\operatorname{div} \mathbf{b})^2 + 102 \operatorname{div} \mathbf{b} + 684(\kappa + \operatorname{div} \mathbf{n}) + 808]\psi$$

$$+ [-2(\operatorname{div} \mathbf{b})^2 + 114 \operatorname{div} \mathbf{b} + 36(\kappa + \operatorname{div} \mathbf{n}) + 8]\,\frac{\delta}{\delta s}(\psi-\theta)$$

$$+ [37(\operatorname{div} \mathbf{b})^2 + 6 \operatorname{div} \mathbf{b} - 12(\kappa + \operatorname{div} \mathbf{n}) - 136]\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right)$$

$$+ [72(\operatorname{div} \mathbf{b})^2 - 324 \operatorname{div} \mathbf{b} - 2640(\kappa + \operatorname{div} \mathbf{n}) - 576]\,\frac{\delta\iota}{\delta s}$$

$$+ 148(\kappa + \operatorname{div} \mathbf{n})\operatorname{div} \mathbf{b} - 37(\operatorname{div} \mathbf{b})^2 - 2598 \operatorname{div} \mathbf{b}$$

$$+ 468(\kappa + \operatorname{div} \mathbf{n}) + 576 = 0 \,. \qquad (5.31)$$

Substituting for $\frac{\delta\iota}{\delta s}$ from (5.11) and then ψ from (5.6), then eliminating $\frac{\delta}{\delta s}(\psi-\theta)$ in favour of $\frac{\delta\kappa}{\delta n}$ by (5.3), we get

$$\left[-\frac{148}{3}(\mathrm{div}\,\underset{\sim}{b})^2 + 332\,\mathrm{div}\,\underset{\sim}{b} + 2000\,(\kappa + \mathrm{div}\,\underset{\sim}{n}) + \frac{1568}{3}\right]\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right)$$

$$+\left[-\frac{259}{3}(\mathrm{div}\,\underset{\sim}{b})^2 + 326\,\mathrm{div}\,\underset{\sim}{b} + 2012\,(\kappa + \mathrm{div}\,\underset{\sim}{n}) + \frac{1976}{3}\right]\frac{\delta}{\delta n}(\psi-\theta)$$

$$+\left[-2(\mathrm{div}\,\underset{\sim}{b})^2 + 114\,\mathrm{div}\,\underset{\sim}{b} + 36(\kappa + \mathrm{div}\,\underset{\sim}{n}) + 8\right]\frac{\delta\kappa}{\delta n}$$

$$-\frac{28663}{18}(\mathrm{div}\,\underset{\sim}{b})^3 + \frac{53033}{9}(\mathrm{div}\,\underset{\sim}{b})^2 + \frac{103072}{9}\,\mathrm{div}\,\underset{\sim}{b}$$

$$-36(\kappa + \mathrm{div}\,\underset{\sim}{n})^2 + 2(\kappa + \mathrm{div}\,\underset{\sim}{n})(\mathrm{div}\,\underset{\sim}{b})^2 + \frac{112396}{3}(\kappa + \mathrm{div}\,\underset{\sim}{n})\,\mathrm{div}\,\underset{\sim}{b}$$

$$+\frac{9968}{3}(\kappa + \mathrm{div}\,\underset{\sim}{n}) + \frac{9680}{9} = 0\,. \qquad (5.32)$$

Using (5.4) to eliminate $\frac{\delta\kappa}{\delta s} + \theta\kappa$ from (5.32) we now obtain*

$$\left[-\frac{407}{3}(\mathrm{div}\,\underset{\sim}{b})^2 + 658\,\mathrm{div}\,\underset{\sim}{b} + 4012\,(\kappa + \mathrm{div}\,\underset{\sim}{n}) + \frac{3544}{3}\right]\frac{\delta}{\delta n}(\psi-\theta)$$

$$+\left[-\frac{74}{3}(\mathrm{div}\,\underset{\sim}{b})^3 + 164\,(\mathrm{div}\,\underset{\sim}{b})^2 + 1000\,(\kappa + \mathrm{div}\,\underset{\sim}{n})\mathrm{div}\,\underset{\sim}{b} + \frac{1126}{3}\,\mathrm{div}\,\underset{\sim}{b}\right.$$

$$\left. + 36(\kappa + \mathrm{div}\,\underset{\sim}{n}) + 8\right]\frac{\delta\kappa}{\delta n}$$

*One also obtained the general form of (5.33) by substituting the expressions (2.14) for $\frac{\delta t}{\delta s}$ and (4.8) for ψ into (2.30) and collecting terms. Then $\frac{\delta\kappa}{\delta s} + \theta\kappa$ was eliminated in favour of $\frac{\delta}{\delta n}(\psi-\theta)$, $\frac{\delta\kappa}{\delta n}$, $\frac{\delta}{\delta n}\,\mathrm{div}\,\underset{\sim}{b}$ and the curvatures using (4.5). Giving the numerical values to $\theta-\psi$, τ, κ and ξ^2 we checked (5.33).

Once again the reduction to (5.34) was checked by the device of setting $\mathrm{div}\,\underset{\sim}{b} = 1$ before and after collecting terms.

$$+ \left[- \frac{74}{3} (\text{div } \underset{\sim}{b})^2 + 166 \text{ div } \underset{\sim}{b} + 1000(\kappa + \text{div } \underset{\sim}{n}) + \frac{784}{3} \right] \frac{\delta}{\delta n} \text{div } \underset{\sim}{b}$$

$$- \frac{3119}{2} (\text{div } \underset{\sim}{b})^3 + 5737 (\text{div } \underset{\sim}{b})^2 + \frac{31984}{3} \text{div } \underset{\sim}{b} - 6036(\kappa + \text{div } \underset{\sim}{n})^2$$

$$+ 150 (\kappa + \text{div } \underset{\sim}{n})(\text{div } \underset{\sim}{b})^2 + 35136(\kappa + \text{div } \underset{\sim}{n})\text{div } \underset{\sim}{b} - 912(\kappa + \text{div } \underset{\sim}{n})$$

$$+ \frac{1136}{3} = 0 . \qquad (5.33)$$

Eliminating $\kappa + \text{div } \underset{\sim}{n}$, $(\kappa + \text{div } \underset{\sim}{n})^2$ and $\frac{\delta}{\delta n} \text{div } \underset{\sim}{b} = \frac{\delta}{\delta b} (\kappa + \text{div } \underset{\sim}{n})$ from (5.33) using (5.7), (5.8) and (5.10) reducing, and dividing out a factor 8 we get

$$56(84) \; [20424(\text{div } \underset{\sim}{b})^3 + 3540(\text{div } \underset{\sim}{b})^2 + 104336 \text{ div } \underset{\sim}{b} + 192] \; \frac{\delta\kappa}{\delta n}$$

$$+ 56(84)[57276(\text{div } \underset{\sim}{b})^2 + 8880 \text{ div } \underset{\sim}{b} + 216128] \; \frac{\delta}{\delta n} (\psi - \theta)$$

$$- 15586065 (\text{div } \underset{\sim}{b})^5 - 22671972(\text{div } \underset{\sim}{b})^4 + 1382504976(\text{div } \underset{\sim}{b})^3$$

$$- 2899136 (\text{div } \underset{\sim}{b})^2 + 8985163008 \text{ div } \underset{\sim}{b} + 915204224 = 0 . \qquad (5.34)$$

This the final relation required to demonstrate the lemma.

We may now eliminate $\frac{\delta}{\delta n} (\psi - \theta)$ between (5.28) and (5.34), to give the expression*

$$56(84)P_A(5) \frac{\delta\kappa}{\delta n} + q_A(7) = 0 \qquad (5.35)$$

*The calculations leading to (5.39) and (5.41) were checked by comparing sums for div b = 1, before and after collecting terms.

where

$$p_A(5) = 20424(\mathrm{div}\ \underset{\sim}{b})^5 + 120890088(\mathrm{div}\ \underset{\sim}{b})^4 + 44029376(\mathrm{div}\ \underset{\sim}{b})^3 + 1474800000(\mathrm{div}\ \underset{\sim}{b})^2 + 250762240\ \mathrm{div}\ \underset{\sim}{b} + 3807307776$$

$$q_A(7) = -15586065(\mathrm{div}\ \underset{\sim}{b})^7 - 153718916544(\mathrm{div}\ \underset{\sim}{b})^6 - 166944059520(\mathrm{div}\ \underset{\sim}{b})^5 - 532743047104(\mathrm{div}\ \underset{\sim}{b})^4 - 1331608168960(\mathrm{div}\ \underset{\sim}{b})^3 + 1438336672768(\mathrm{div}\ \underset{\sim}{b})^2 - 2693568049152\ \mathrm{div}\ \underset{\sim}{b} + 2033701011456 .$$

Again, eliminating $\frac{\delta}{\delta n}(\phi=0)$ between (5.17) and (5.28), we obtain

$$6(56)^2\ p_B(5)\ \frac{\delta\kappa}{\delta n} + q_B(7) = 0 \tag{5.36}$$

where

$$p_B(5) = -2220(\mathrm{div}\ \underset{\sim}{b})^5 - 13120440(\mathrm{div}\ \underset{\sim}{b})^4 - 4153856(\mathrm{div}\ \underset{\sim}{b})^3 - 157726848(\mathrm{div}\ \underset{\sim}{b})^2 - 21606400\ \mathrm{div}\ \underset{\sim}{b} - 394598400$$

and

$$q_B(7) = 6776550(\mathrm{div}\ \underset{\sim}{b})^7 + 66732490488(\mathrm{div}\ \underset{\sim}{b})^6 + 69323045568(\mathrm{div}\ \underset{\sim}{b})^5 + 218664779392(\mathrm{div}\ \underset{\sim}{b})^4 + 562821139456(\mathrm{div}\ \underset{\sim}{b})^3 - 633885792256(\mathrm{div}\ \underset{\sim}{b})^2 + 1125361999872\ \mathrm{div}\ \underset{\sim}{b} - 842975477760 .$$

The remaining condition (5.24) can be written

$$56(84)(4)\ p_C(2)\ \frac{\delta\kappa}{\delta n} + q_C(4) = 0 , \tag{5.37}$$

so eliminating $56(84)\,\frac{\delta\kappa}{\delta n}$ between (5.35) and (5.37) we get

$$p_A(5)\; q_c(4) - 4\; p_c(2)\; q_A(7) = 0$$

or

$$\left[\frac{p_A(5)}{8}\right]\left[\frac{q_c(4)}{3}\right] - \left[\frac{p_c(2)}{6}\right] q_A(7) = 0\;. \tag{5.38}$$

Writing $x = \operatorname{div} \underset{\sim}{b}$ we obtain as the expanded form of (5.38),

$$\begin{aligned}
&[2553x^5 + 15111261x^4 + 5503672x^3 + 184350000x^2 + 31345280x + 475913472] \\
&\quad \times [1556627x^4 + 1328012x^3 + 1805184x^2 + 5944256x - 3550976] \\
&\quad + \left[111x^2 + 12x + 912\right] \\
&\quad \times \begin{bmatrix} -155860065x^7 \quad 155718916544x^6 - 166944059520x^5 \\ -532743047104x^4 - 1331608168960x^3 + 1438336672768x^2 \\ -\; 2693568049152x + 2035701011156 \end{bmatrix} = 0
\end{aligned} \tag{5.39}$$

The coefficient of $\frac{\delta\kappa}{\delta n}$ in (5.37) is equal to $6(56)^2\; p_c\;(2)\,\frac{\delta\kappa}{\delta n}$, so we may eliminate $6(56)^2\,\frac{\delta\kappa}{\delta n}$ between (5.36) and (5.37) to obtain

$$p_B(5)\; q_c(4) - q_B(7)\; p_c(2) = 0$$

or

$$\left[\frac{p_B(5)}{2}\right]\left[\frac{q_c(4)}{3}\right] - q_{B(7)}\;\left[\frac{p_c(2)}{6}\right] = 0\;. \tag{5.40}$$

With $x = \operatorname{div} b$ the expanded form of (5.40) is

$$\left[-1110x^5 - 6560220x^4 - 2076928x^3 - 7886342x^2 - 10803200x - 197299200\right]$$

$$\times \left[1556627x^4 + 1328012x^3 + 1805184x^2 + 5944256x - 3550976\right]$$

$$+ (111x^2 + 12x + 912)$$

$$\times \begin{bmatrix} 6776550x^7 + 66752490488x^6 + 69323045568x^5 \\ + 218664779392x^4 + 562821139456x^3 \\ - 633885792256x^2 + 1125361999872x - 842975477760 \end{bmatrix} = 0 \,. \quad (5.41)$$

Now the conditions (5.39) and (5.41) are quite close. Using a hand calculator dividing out the coefficient of x^9 and approximating to five significant figures, we obtain for them, repsectively

$$x^9 + 2880.0x^8 + 3676.0x^7 + 55300x^6 + 38720x^5 + 335100x^4 + 11970x^3$$

$$+ 84400x^2 + 12720x + 73400 = 0 \qquad (5.39a)$$

and

$$x^9 + 2875.9x^8 + 3551.1x^7 + 52686x^6 + 36866x^5 + 324970x^4 + 11512x^3$$

$$+ 81842x^2 + 12117x + 69890 = 0 \,. \qquad (5.41b)$$

We therefore expect the roots of (5.39) to be quite close to those of (5.41). We use the Georgia Institute of Technology Cyber 170/650 Computer programmed to double precision to accommodate thirty digits, for multiplying out the terms exactly. The exact values of the coefficients were then automatically entered into a double precision program for obtaining the roots.*

Equation (5.39) is written

$$\sum_{r=0}^{9} a_r x^r = 0 \tag{5.42}$$

where

$$\begin{aligned}
a_0 &= 164778005299200. \\
a_1 &= -285515525734400. \\
a_2 &= 189599436939673 6. \\
a_3 &= 268730322651136. \\
a_4 &= 75210226242048 0. \\
a_5 &= 8690879462924 8. \\
a_6 &= 1202374553417 44. \\
a_7 &= 82500769157 00. \\
a_8 &= 6463000522119. \\
a_9 &= 2244015516.
\end{aligned} \tag{5.43}$$

The nine roots of (5.42) are**

*As independent checks on (5.43) and (5.44) the writer calculated the coefficients a_7, a_8, a_9, and b_7, b_8, b_9, exactly, using a Friden calculator. Also Professor J. T. S. Wang obtained all the coefficients to an accuracy of eight digits using an Apple Computer.

**While the coefficients are exact, the computed roots have been truncated, after computation, to twelve decimal places in both real and imaginary parts.

$$
\begin{aligned}
&-2878.834186924203 + 0.000000000000i \\
&\quad 0.623093217621 - 2.118683285446i \\
&\quad 0.623093217621 + 2.118683285446i \\
&- 0.870533152045 - 2.390229527263i \\
&- 0.870533152045 + 2.390229527263i \\
&- 0.312686810106 - 2.960102978562i \\
&- 0.312686810106 + 2.960102978562i \\
&- 0.075177567060 - 0.292521445547i \\
&- 0.075177567060 + 0.292521445547i
\end{aligned}
\tag{5.44}
$$

Equation (5.41) is written

$$\sum_{r=0}^{9} b_r x^r = 0 \tag{5.45}$$

where

$$
\begin{aligned}
b_0 &= -6818891169792 0. \\
b_1 &= -1182206113218 56. \\
b_2 &= -7985059961405 44. \\
b_3 &= -1123252971274 24. \\
b_4 &= -3170668306391 04. \\
b_5 &= -35969307556864. \\
b_6 &= 514044245382 40. \\
b_7 &= -3445228681232. \\
b_8 &= -280590190849 2. \\
b_9 &= -975658920.
\end{aligned}
\tag{5.46}
$$

The nine roots of (5.45) are

$$
\begin{array}{l}
-2874.682532156276 + 0.000000000000i \\
0.641796843209 - 2.111826154258i \\
0.641796843209 + 2.111826154258i \\
-0.872480050191 - 2.345494413109i \\
-0.872480050191 + 2.345494413109i \\
-0.306461021321 - 2.967161661867i \\
-0.306461021321 + 2.967161661867i \\
-0.073855067199 - 0.290004746692i \\
-0.073855067199 + 0.290004746692i
\end{array}
\tag{5.47}
$$

Comparing the lists (5.44) and (5.47) we see that the roots of (5.39) are clearly distinct from the roots of (5.41).*

We conclude that the eliminant of x from (5.39) and (5.4) does not vanish for the particular values (5.1). Thus it does not vanish identically. It is thus possible to eliminate x from these equations and obtain a homogeneous polynomial in the variables $\theta-\phi$, τ, and κ. Thus there exist a relation $f\left(\frac{\kappa}{\tau}, \frac{\theta-\phi}{\tau}\right) = 0$. This proves Lemma 1.

*The reader might note for comparison the lists of roots of equations (7.15) and (7.19) where the three common roots are evident.

6. ADDITIONAL CONDITIONS FROM LEMMA 1

With the existence of the functional relation $f\left(\frac{\kappa}{\tau}, \frac{\theta-\psi}{\tau}\right) = 0$ established, we may now assert that

$$\begin{vmatrix} \frac{\delta}{\delta b}\left(\frac{\kappa}{\tau}\right) & \frac{\delta}{\delta b}\left(\frac{\theta-\psi}{\tau}\right) \\ \frac{\delta}{\delta n}\left(\frac{\kappa}{\tau}\right) & \frac{\delta}{\delta n}\left(\frac{\theta-\psi}{\tau}\right) \end{vmatrix} = 0 , \tag{6.1}$$

and

$$\begin{vmatrix} \frac{\delta}{\delta b}\left(\frac{\kappa}{\tau}\right) & \frac{\delta}{\delta b}\left(\frac{\theta-\psi}{\tau}\right) \\ \frac{\delta}{\delta s}\left(\frac{\kappa}{\tau}\right) & \frac{\delta}{\delta s}\left(\frac{\theta-\psi}{\tau}\right) \end{vmatrix} = 0 . \tag{6.2}$$

Expanding (6.1) using (2.10), (2.11) and (2.25) we get, since τ and κ do not vanish,

$$\operatorname{div} \underset{\sim}{b} \frac{\delta}{\delta n}(\psi-\theta) - 4\tau \frac{\delta\tau}{\delta n} = 0 . \tag{6.3}$$

Expressing $\frac{\delta\tau}{\delta n}$ in terms of $\frac{\delta}{\delta s}(\psi-\theta)$ by (2.31) we transform (6.3) to

$$\operatorname{div} \underset{\sim}{b} \frac{\delta}{\delta n}(\psi-\theta) - 4\tau \frac{\delta}{\delta s}(\psi-\theta) - 4\tau\left[\psi^2-\theta^2-12\tau^2+\kappa^2+\kappa(\kappa+\operatorname{div}\underset{\sim}{n})\right] = 0 . \tag{6.4}$$

From (4.2) and (6.4) we obtain

$$2(\theta-\psi)\left[\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + (\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b}\right]$$

$$+ \tau\Big\{-24(\operatorname{div}\underset{\sim}{b})^2 + 8\kappa(\kappa + \operatorname{div}\underset{\sim}{n}) + 150\tau^2 - 60\psi$$

$$+ \frac{72\tau}{\kappa}\left[(\theta-\psi)\operatorname{div}\underset{\sim}{b} + 6\tau(\kappa + \operatorname{div}\underset{\sim}{n})\right]\Big\} = 0 \quad . \tag{6.5}$$

Expanding (6.2), using (2.10), (2.11) and (2.25) we similarly obtain

$$\tau \operatorname{div}\underset{\sim}{b}\,\frac{\delta}{\delta s}(\psi-\theta) - 4\tau^2\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) + 4\tau^2\theta\kappa$$

$$+ \left[(\theta-\psi)\operatorname{div}\underset{\sim}{b} + 4\tau\kappa\right]\frac{\delta\tau}{\delta s} = 0 \ . \tag{6.6}$$

Eliminating $\operatorname{div}\underset{\sim}{b}\,\frac{\delta}{\delta s}(\psi-\theta) - 4\tau\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right)$ from (6.6) using (4.4) and using the expression (2.14) for $\frac{\delta\tau}{\delta s}$ we get

$$-\tau\kappa\left[\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + (\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b}\right] - \frac{1}{3}(\theta-\psi)\kappa\,(\operatorname{div}\underset{\sim}{b})^2$$

$$+ 6\tau^2(\theta-\psi)(\kappa + \operatorname{div}\underset{\sim}{n}) + \tau\left[-\frac{8\kappa^2}{3} + \frac{7}{3}(\psi-\theta)^2 + 48\tau^2\right]\operatorname{div}\underset{\sim}{b}$$

$$+ \frac{\tau^2\kappa}{3}(14\theta - 44\psi) = 0 \ . \tag{6.7}$$

We now employ the subsidiary variable λ defined in (2.36) to eliminate ψ in favour of $\theta-\psi$. We have

$$\psi = \frac{(\lambda - \kappa\operatorname{div}\underset{\sim}{b})}{6\tau} - \frac{1}{3}(\theta-\psi) \tag{6.8}$$

Then

$$\frac{\tau^2\kappa}{3}\,(14\theta - 44\psi) = \frac{14\tau^2\kappa}{3}\,(\theta-\psi) - 10\tau^2\kappa\psi$$

$$= 8\tau^2\kappa(\theta-\psi) - \frac{5}{3}\,\tau\kappa(\lambda - \kappa \operatorname{div} \underset{\sim}{b}) \quad \text{, by (6.8)} \ ,$$

and (6.7) takes the form

$$-\tau\kappa\left[\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + (\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b}\right] - \frac{1}{3}\,(\theta-\psi)\kappa(\operatorname{div}\underset{\sim}{b})^2$$

$$+ 6\tau^2(\theta-\psi)(\kappa + \operatorname{div}\underset{\sim}{n}) + \tau\left[-\kappa^2 + \frac{7}{3}(\psi-\theta)^2 + 48\tau^2\right]\operatorname{div}\underset{\sim}{b}$$

$$+ 8\tau^2\kappa(\theta-\psi) - \frac{5}{3}\,\tau\kappa\lambda = 0 \ . \qquad (6.9)$$

Eliminating the expression $\frac{\delta}{\delta n}\operatorname{div}\underset{\sim}{b} + (\kappa + \operatorname{div}\underset{\sim}{n})\operatorname{div}\underset{\sim}{b}$ from (6.5) and (6.9) by multiplying the former by $\tau\kappa$ and the latter by $2(\theta-\psi)$ and adding we get, on reduction

$$12\tau^2\left[\xi^2 + \frac{2\kappa^2}{3}\right](\kappa + \operatorname{div}\underset{\sim}{n}) - \frac{2}{3}\,\kappa\xi^2(\operatorname{div}\underset{\sim}{b})^2 + \left(\frac{14}{3}\,\xi^2 - 2\kappa^2\right)(\theta-\psi)\tau\operatorname{div}\underset{\sim}{b}$$

$$+ \tau^2\kappa\,[15\theta^2 - 6\theta\psi] + 16\tau^2\kappa(\theta-\psi)^2 - \frac{10}{3}\,\tau\kappa\lambda(\theta-\psi) = 0 \ . \qquad (6.10)$$

Again by (6.8)

$$6\theta\psi = 6(\theta-\psi)\psi + 6\psi^2$$

$$= (\theta-\psi)\left[\frac{\lambda}{\tau} - \frac{\kappa\operatorname{div}\underset{\sim}{b}}{\tau} - 2(\theta-\psi)\right] + \frac{1}{6\tau^2}\,[\lambda - \kappa\operatorname{div}\underset{\sim}{b} - 2\tau(\theta-\psi)]^2$$

$$= \frac{1}{3}\left(\frac{\theta-\psi}{\tau}\right)(\lambda - \kappa\operatorname{div}\underset{\sim}{b}) - \frac{1}{3}(\theta-\psi)^2 + \frac{1}{6\tau^2}\,(\lambda - \kappa\operatorname{div}\underset{\sim}{b})^2 \ . \qquad (6.11)$$

Eliminating $-6\tau^2\kappa\theta\psi$ from (6.10) we get

$$12\tau^2\left[\xi^2+\frac{2\kappa^2}{3}\right](\kappa+\operatorname{div}\underset{\sim}{n})-\frac{2}{3}\kappa\xi^2(\operatorname{div}\underset{\sim}{b})^2$$

$$+\frac{1}{3}(14\xi^2-5\kappa^2)(\theta-\psi)\tau\operatorname{div}\underset{\sim}{b}+150\tau^4\kappa+\frac{52}{3}\tau^2\kappa(\theta-\psi)^2$$

$$-\frac{11\lambda\tau\kappa}{3}(\theta-\psi)-\frac{\kappa}{6}(\lambda-\kappa\operatorname{div}\underset{\sim}{b})^2=0\,. \tag{6.12}$$

From (4.14) we have

$$-12\tau^2\left[\xi^2+\frac{\kappa^2}{3}\right](\kappa+\operatorname{div}\underset{\sim}{n})+\frac{2}{3}\kappa\xi^2(\operatorname{div}\underset{\sim}{b})^2$$

$$-2(\xi^2-\kappa^2)(\theta-\psi)\tau\operatorname{div}\underset{\sim}{b}-\frac{8}{3}\tau^2\kappa(\xi^2-\kappa^2)=0\,,$$

and by adding this to (6.12) we get (6.13)

$$4\tau^2\kappa^2(\kappa+\operatorname{div}\underset{\sim}{n})+\frac{1}{3}(8\xi^2+\kappa^2)(\theta-\psi)\tau\operatorname{div}\underset{\sim}{b}$$

$$+\kappa\left[\frac{44}{3}\tau^2\xi^2+\frac{8}{3}\tau^2\kappa^2\quad 474\,\tau^4\right]$$

$$-\kappa\left[\frac{11}{3}\lambda\tau(\theta-\psi)+\frac{1}{6}(\lambda-\kappa\operatorname{div}\underset{\sim}{b})^2\right]=0\,. \tag{6.14}$$

Finally on multiplying (6.14) by $3(3\xi^2+\kappa^2)$ and (6.13) by $3\kappa^2$ and adding to eliminate $(\kappa+\operatorname{div}\underset{\sim}{n})$ we get

$$\frac{1}{2}(\xi^2 - \kappa^2)\kappa^3(\mathrm{div}\ \underset{\sim}{b})^2 + [(7\kappa^4 + 5\xi^2\kappa^2 + 24\xi^4)(\theta-\psi)\tau + \lambda\kappa^2(3\xi^2 + \kappa^2)]\mathrm{div}\ \underset{\sim}{b}$$

$$+ \kappa\left[-(3\xi^2 + \kappa^2)\left[11\lambda\tau(\theta-\psi) + \frac{\lambda^2}{2}\right] + \tau^2[16\kappa^4 + 60\xi^2\kappa^2 + 132\xi^4 - 1422\tau^2(3\xi^2 + \kappa^2)\right] =$$

(6.15)

7. LEMMA 2

We prove

LEMMA 2. There exist a functional relation $F(\theta-\psi,\lambda,\tau) = 0$.

We prove Lemma 2 by obtaining three polynomial relations among the variables κ, div $\underset{\sim}{b}$, $\theta-\psi,\lambda$ and τ and showing that κ and div $\underset{\sim}{b}$ may be eliminated from these. This eliminant set equal to zero is the required relation. We remember that ξ^2 is given in terms of $\theta-\psi$ and τ by (2.38). The three conditions are the following:

1) The condition (6.15).

2) The condition obtained by eliminating κ + div $\underset{\sim}{n}$ and $\frac{\delta}{\delta \underset{\sim}{b}}(\kappa + \text{div } \underset{\sim}{n}) = \frac{\delta}{\delta \underset{\sim}{n}} \text{div } \underset{\sim}{b}$ from (6.9) using (4.14) and (4.17).

3) The condition obtained by taking the derivative of (6.15) with respect to $\underset{\sim}{b}$.

We perform the elimination with the numerical values

$$\theta-\psi = \tau = \lambda = 1 , \quad \xi^2 = 37 , \tag{7.1}$$

and complicated functions of $\theta-\psi$, τ, and λ are represented by numbers.

With these values the condition (6.15) multiplied by 2 becomes

$$a(\text{div } \underset{\sim}{b})^2 + b(\text{div } \underset{\sim}{b}) + c = 0 ,$$

where

$$
\begin{aligned}
a &= (37 - \kappa^2)\kappa^3 \ , \\
b &= 16\kappa^4 + 592\kappa^2 + 65712 \ , \\
c &= (32\kappa^4 + 1573\kappa^2 + 43179)\kappa \ . \qquad (7.2)
\end{aligned}
$$

The basic integral (4.14) and its gradient with respect to $\underset{\sim}{b}$ given by (4.17) become respectively

$$
\begin{aligned}
6(\kappa^2 + 111)(\kappa + \operatorname{div} \underset{\sim}{n}) = 37\kappa(\operatorname{div} \underset{\sim}{b})^2 - (111 - 3\kappa^2)\operatorname{div} \underset{\sim}{b} \\
- 4\kappa(37 - \kappa^2) \ , \qquad (7.3)
\end{aligned}
$$

and

$$
\begin{aligned}
6(\kappa^2 + 111)\ \frac{\delta}{\delta \underset{\sim}{b}}\ (\kappa + \operatorname{div} \underset{\sim}{n}) &= 12(\kappa^2 - 222)(\kappa + \operatorname{div} \underset{\sim}{n})\operatorname{div} \underset{\sim}{b} \\
&- 288\kappa(\kappa + \operatorname{div} \underset{\sim}{n}) + 37\kappa(\operatorname{div} \underset{\sim}{b})^3 + (5\kappa^2 - 333)(\operatorname{div} \underset{\sim}{b})^2 \\
&- 912\kappa \operatorname{div} \underset{\sim}{b} - 32\kappa^2 \ . \qquad (7.4)
\end{aligned}
$$

On eliminating $(\kappa + \operatorname{div} \underset{\sim}{n})$ from (7.4) using (7.3), we get on cancelling the factor 6,

$$
\begin{aligned}
6(\kappa^2 + 111)^2\ \frac{\delta}{\delta \underset{\sim}{b}}\ (\kappa + \operatorname{div} \underset{\sim}{n}) &= 6(\kappa^2 + 111)^2\ \frac{\delta}{\delta n}\operatorname{div} \underset{\sim}{b} \\
&= 37\kappa(3\kappa^2 - 333)(\operatorname{div} \underset{\sim}{b})^3 + [11\kappa^4 - 1332\kappa^2 + 10545](\operatorname{div} \underset{\sim}{b})^2 \\
&+ 8\kappa[\kappa^4 - 391\kappa^2 - 3774](\operatorname{div} \underset{\sim}{b}) \\
&+ 32\kappa^2[-7\kappa^2 + 111] \ . \qquad (7.5)
\end{aligned}
$$

For the particular values (7.1), the condition (6.9) reduces to

$$- \kappa\left[\frac{\delta}{\delta n}\,\mathrm{div}\,\underset{\sim}{b} + (\kappa + \mathrm{div}\,\underset{\sim}{n})\mathrm{div}\,\underset{\sim}{b}\right] - \frac{1}{3}\,(\mathrm{div}\,\underset{\sim}{b})^2 + \left(-\kappa^2 + \frac{151}{3}\right)\mathrm{div}\,\underset{\sim}{b}$$

$$+ 6(\kappa + \mathrm{div}\,\underset{\sim}{b}) + \frac{19}{3} = 0\ . \qquad (7.6)$$

Multiplying (7.6) by $6(\kappa^2 + 111)^2$ we eliminate $(\kappa + \mathrm{div}\,\underset{\sim}{n})$ and $\frac{\delta}{\delta n}\,\mathrm{div}\,\underset{\sim}{b}$ using the expressions (7.3) and (7.5). On reduction we obtain*

$$2(37)\kappa^2(111 - 2\kappa^2)(\mathrm{div}\,\underset{\sim}{b})^3 + \kappa^3[-16\kappa^2 + 37(72)](\mathrm{div}\,\underset{\sim}{b})^2$$

$$+ [-18\kappa^6 + 1820\kappa^4 + 30(37)^2\kappa^2 + 72(37)^3](\mathrm{div}\,\underset{\sim}{b})$$

$$+ \kappa[286\kappa^4 + 6660\kappa^2 + 369630] = 0\ . \qquad (7.7)$$

It remains to obtain the gradient of the relation (6.15) with respect to $\underset{\sim}{b}$.

Eliminating $\kappa + \mathrm{div}\,\underset{\sim}{n}$ from (2.29) using (4.14) we get**

---

*One also obtained the unspecialized form of (7.7) by eliminating $(\kappa + \mathrm{div}\,\underset{\sim}{n})$ and $\frac{\delta}{\delta b}(\kappa + \mathrm{div}\,\underset{\sim}{n})$ from (6.9) using (4.14) and (4.17). On substituting the values (7.1) one checked the form (7.7).

**The condition shows immediately that if $\mathrm{div}\,\underset{\sim}{b} = 0$ then either τ or κ must be zero, so that there are no solutions for this case.

$$(3\xi^2 + \kappa^2)\ \frac{\delta}{\delta \underset{\sim}{b}}\,\mathrm{div}\ \underset{\sim}{b} = -\ (3\xi^2 - \kappa^2)(\mathrm{div}\ \underset{\sim}{b})^2 - 24\kappa(\theta-\psi)\tau\ \mathrm{div}\ \underset{\sim}{b}$$

$$- 32\tau^2\kappa^2\ . \tag{7.8}$$

For the particular values (7.1) we have

$$(111 + \kappa^2)\ \frac{\delta}{\delta \underset{\sim}{b}}\,\mathrm{div}\ \underset{\sim}{b} = -\ (111 - \kappa^2)(\mathrm{div}\ \underset{\sim}{b})^2 - 24\kappa\ \mathrm{div}\ \underset{\sim}{b}$$

$$- 32\kappa^2\ , \tag{7.9}$$

and also by (2.11), (2.25) and (2.39)

$$\frac{\delta\kappa}{\delta \underset{\sim}{b}} = -\ \kappa\ \mathrm{div}\ \underset{\sim}{b}\ , \quad \frac{\delta}{\delta \underset{\sim}{b}}\,(\theta-\psi) = 4\kappa\ , \quad \frac{\delta\xi^2}{\delta \underset{\sim}{b}} = 8\kappa\ , \tag{7.10}$$

Taking the gradient of (6.15) and inserting the numerical values we get

$$[(37 - \kappa^2)\kappa^3\mathrm{div}\ \underset{\sim}{b} + (8\kappa^4 + 296\kappa^2 + 32856)]\ \frac{\delta}{\delta \underset{\sim}{b}}\,\mathrm{div}\ \underset{\sim}{b}$$

$$+ \left[\frac{1}{2}\ (111\kappa^2 - 5\kappa^4)(\mathrm{div}\ \underset{\sim}{b})^2 + (32\kappa^3 + 592\kappa)\mathrm{div}\ \underset{\sim}{b} + 80\kappa^4\right.$$

$$\left. + \frac{4719}{2}\,\kappa^2 + \frac{43179}{2}\right]\ \frac{\delta\kappa}{\delta \underset{\sim}{b}}$$

$$+\ [(7\kappa^4 + 185\kappa^2 + 32856)\mathrm{div}\ \underset{\sim}{b} - (1221 + 11\kappa^2)\kappa]\ \frac{\delta}{\delta \underset{\sim}{b}}\,(\theta-\psi)$$

$$+ \left[\frac{\kappa^3(\mathrm{div}\ \underset{\sim}{b})^2}{2} + (8\kappa^2 + 1776)\mathrm{div}\ \underset{\sim}{b} + \kappa\left(60\kappa^2 + \frac{10935}{2}\right)\right]\frac{\delta\xi^2}{\delta \underset{\sim}{b}} = 0\ .$$

Substituting the values (7.9) and (7.10) we obtain the expression

$$-\kappa^3 \left[\frac{20535}{2} - 370\kappa^2 - \frac{3}{2}\kappa^4\right] (\mathrm{div}\ \underset{\sim}{b})^3$$

$$+ [4\kappa^6 - 5180\kappa^4 - 65712\kappa^2 - 3647016] (\mathrm{div}\ \underset{\sim}{b})^2$$

$$+ \kappa \left[-20\kappa^6 - \frac{17407}{2}\kappa^4 - 55724\kappa^2 + \frac{25960347}{2}\right] \mathrm{div}\ \underset{\sim}{b}$$

$$+ 180\kappa^6 + 77780\kappa^4 + 3261624\kappa^2 = 0 \ . \tag{7.11}$$

By means of the integral (7.2) we may now reduce (7.7) and (7.11) to quadratics in div $\underset{\sim}{b}$. Multiplying (7.7) by $\kappa(37 - \kappa^2)$ and (7.2) by $2(37)(111 - 2\kappa^2)$div $\underset{\sim}{b}$ and subtracting we obtain the form*

$$a_1 (\mathrm{div}\ \underset{\sim}{b})^2 + b_1\ \mathrm{div}\ \underset{\sim}{b} + c_1 = 0$$

where

$$a_1 = 8[2\kappa^8 - 111\kappa^6 + 6845\kappa^4 + 607836\kappa^2 - 67469796]$$

$$b_1 = [18\kappa^8 + 2250\kappa^6 - 3774\kappa^4 - 8657556\kappa^2 - 219732714]\kappa$$

$$c_1 = [-280\kappa^6 + 3922\kappa^4 - 123210\kappa^2 + 13676310]\kappa^2 \ . \tag{7.12}$$

Multiplying (7.11) by $(37 - \kappa^2)$ and (7.2) by $\left[\frac{20535}{2} - 370\kappa^2 - \frac{3}{2}\kappa^4\right]$ div $\underset{\sim}{b}$ and adding we obtain

*The numerical calculations leading to (7.12), (7.13), (7.14) and (7.15) were checked by comparing values for $\kappa = 1$ before and after multiplying out and collecting terms.

$$a_2(\mathrm{div}\ \underline{b})^2 + b_2\ \mathrm{div}\ \underline{b} + c_2 = 0$$

where

$$a_2 = -28\kappa^8 - 1480\kappa^6 - 279276\kappa^4 - 17019408\kappa^2 + 539758368\ ,$$

$$b_2 = [-28\kappa^8 - 6256\kappa^6 - 584526\kappa^4 - 14867340\kappa^2 + 923606802]\kappa\ ,$$

$$c_2 = [-180\kappa^6 - 71120\kappa^4 - 383764\kappa^2 + 120680088]\kappa^2\ . \tag{7.13}$$

Eliminating div $\underline{b}$ from (7.2) and (7.12) we have, writing $x = \kappa^2$.

$$(ca_1 - c_1a)^2 - (bc_1 - b_1c)(ab_1 - a_1b) = 0$$

or, since κ does not vanish,

$$\begin{aligned}
&[226x^6 + 11256x^5 + 778036x^4 + 221635624x^3 - 7764800982x^2 \\
&\qquad - 639073907712x - 23306226571872]^2 \\
&- [-576x^6 - 101890x^5 - 4302264x^4 + 167382376x^3 + 21216343086x^2 \\
&\qquad + 719464169646x + 10386536540526] \\
&\times [-18x^7 - 1840x^6 + 91760x^5 + 7116062x^4 - 152465530x^3 - 5971076946x^2 \\
&\qquad + 0x + 35468601878016] = 0\ .
\end{aligned} \tag{7.14}$$

Multiplying out equation (7.14) we obtain the equation

$$\sum_{r=0}^{13} c_r x^r = 0 \tag{7.15}$$

where the coefficients c_r have the following values*

$$\begin{aligned}
c_0 &= 1747842675720501858891079 68.\\
c_1 &= 427041437994363970425139 2.\\
c_2 &= 79856664123448307747772.\\
c_3 &= -463670866717486786008.\\
c_4 &= 5580483308298759031 2.\\
c_5 &= -30793323482538675 60.\\
c_6 &= -1655042956088577 16.\\
c_7 &= -30281781161261 04.\\
c_8 &= 49898258252800.\\
c_9 &= 1860931538408.\\
c_{10} &= 9298644928.\\
c_{11} &= -212496880.\\
c_{12} &= -2896784.\\
c_{13} &= -10368.
\end{aligned} \tag{7.16}$$

The thirteen roots of (7.15) are as follows:

$$\begin{array}{r}
11.892723470319 -27.263198465047i\\
11.892723470319 +27.263198465047i\\
-71.786195400710 -47.265501777280i\\
-71.786195400710 +47.265501777280i\\
-110.999996942690 + 0.000000000000i\\
-111.000003057309 + 0.000000000000i\\
36.999999999998 + 0.000000000000i\\
57.119373083167 + 0.000000000000i\\
61.925918935479 + 0.000000000000i\\
-22.058046586495 -29.530326668773i\\
-22.058046586495 +29.530326668773i\\
-24.769429961572 -23.745986810882i\\
-24.769429961572 +23.745986810882i
\end{array} \tag{7.17}$$

*The same checking procedure was used here as for equations (5.42), (5.43). See footnote following equations (5.42), (5.43).

Eliminating div $\underset{\sim}{b}$ from (7.2) and (7.13) we have, writing $x = \kappa^2$,

$$(ca_2 - c_2a)^2 - (bc_2 - b_2c)(ab_2 - a_2b) = 0$$

or since κ does not vanish,

$$\begin{aligned}
&[-1076x^6 - 155864x^5 - 10226208x^4 - 912947768x^3 - 2602328266 8x^2 \\
&\qquad + 114158894832x + 23306226571872]^2 \\
&- [896x^6 + 240716x^5 + 28478592x^4 + 1604407098x^3 + 16100411990x^2 \\
&\qquad - 764651913558x - 31950288160902] \\
&\times [28x^7 + 5648x^6 + 394050x^5 + 424590x^4 - 938802702x^3 + 53964591834x^2 \\
&\qquad + 798842384640x - 35468601878016] = 0 \quad .
\end{aligned} \tag{7.18}$$

Multiplying out equation (7.18) we obtain the equation

$$\sum_{r=0}^{13} d_r x^r = 0 \tag{7.19}$$

where the coefficients d_r have the following values

$$
\begin{aligned}
d_0 &= -5900518553647488570802946048. \\
d_1 &= 3723336223436505875535360. \\
d_2 &= 1706102929834034426514460. \\
d_3 &= 6817183219413156345192. \\
d_4 &= -1852622228437909179321. \\
d_5 &= -34847821547455514336. \\
d_6 &= 1563776415827778932. \\
d_7 &= 37756479512765448. \\
d_8 &= -91026116850404. \\
d_9 &= -14843023280912. \\
d_{10} &= -254704493488. \\
d_{11} &= -2174614016. \\
d_{12} &= -10642880. \\
d_{13} &= -25088.
\end{aligned}
\tag{7.20}
$$

The thirteen roots of (7.19) are as follows:

$$
\begin{array}{rcr}
-68.598221002871 & - & 121.180161857876i \\
-68.598221002871 & + & 121.180161857876i \\
-110.999990761599 & + & 0.000000000000i \\
-111.000009238403 & + & 0.000000000000i \\
-52.204239222463 & - & 57.522416326431i \\
-52.204239222463 & + & 57.522416326431i \\
-22.372509683479 & - & 23.424162173390i \\
-22.372509683479 & + & 23.424162173390i \\
24.680509035078 & - & 14.636662025164i \\
24.680509035078 & + & 14.636662025164i \\
37.000000000000 & + & 0.000000000000i \\
21.503374813395 & + & 0.000000000000i \\
-23.736391841428 & + & 0.000000000000i
\end{array}
\tag{7.21}
$$

Equations (7.15) and (7.19) have the common roots

$$x = \kappa^2 = -111 \tag{7.22}$$

occurring twice, and

$$x = \kappa^2 = 37 \tag{7.23}$$

occurring once.

The question is, do these common roots result from our particular numerical substitution or do they correspond to common factors of the general polynomial? If the former were true our analysis would tell us nothing. Fortunately we are able to identify these roots as factors of the general polynomial and then demonstrate that they may be discounted. We consider the two cases in detail.

Case 1. The root $\kappa^2 = -111$

We would suspect that this root corresponds to a factor $\kappa^2 + 3\xi^2$. From equation (6.15) multiplied by 2 we have for $\kappa^2 = -3\xi^2$

$$a = 4\xi^2\kappa^2 = -12\kappa\xi^4$$

$$b = 144(\theta-\psi)\tau\xi^4$$

$$c = 192\kappa\tau^2\xi^4 \quad . \tag{7.24}$$

Our procedure was to eliminate $\frac{\delta}{\delta b}(\kappa + \text{div}\,\underset{\sim}{n})\left(= \frac{\delta}{\delta n}\text{div}\,\underset{\sim}{b}\right)$ from (4.17) and (6.9) by multiplying (4.17) by $(3\xi^2 + \kappa^2)\kappa$ and (6.9) by $6(3\xi^2 + \kappa^2)^2\tau$ and then eliminating $6(3\xi^2 + \kappa^2)\tau^2(\kappa + \text{div}\,\underset{\sim}{n})$ in favor of div $\underset{\sim}{b}$, ξ^2, κ, $(\theta-\psi)$ using (4.14). We get

$$2\kappa^2\xi^2(\kappa^2 - 6\xi^2)(\text{div}\,\underset{\sim}{b})^3 - [6(\kappa^2 - 6\xi^2)(\xi^2 - \kappa^2) + 48\kappa^2\xi^2](\theta-\psi)\kappa\tau(\text{div}\,\underset{\sim}{b})^2$$

$$+ [-8\tau^2\xi^2(\kappa^2 - 6\xi^2)(\xi^2 - \kappa^2) + 144\tau^2\kappa^2(\xi^2 - \kappa^2)(\theta-\psi)^2]\text{div}\,\underset{\sim}{b}$$

$$+ 192\tau^3\kappa^3(\theta-\psi)(\xi^2 - \kappa^2) + (3\xi^2 + \kappa^2)^2 p_1 + (3\xi^2 + \kappa^2)p_2 = 0 \tag{7.25}$$

where p_1 and p_2 are polynomials. The cubic (7.25) corresponds to the cubic (7.7).

Equation (6.15) is written

$$(\xi^2 - \kappa^2)\kappa^3(\operatorname{div} \underset{\sim}{b})^2 + 2[(3\kappa^4 + 5\xi^2\kappa^2 + 24\xi^4)(\theta - \psi)\tau]\operatorname{div} \underset{\sim}{b}$$

$$+ 2\kappa\tau^2[10\kappa^4 + 60\xi^2\kappa^2 + 132\xi^4] + (3\xi^2 + \kappa^2)p_3 = 0 \ , \qquad (7.26)$$

where p_3 is a polynomial.

We reduce (7.25) to a quadratic by multiplying (7.25) by $(\xi^2 - \kappa^2)$ and (7.26) by $2\xi^2(\kappa^2 - \xi^2)\operatorname{div} \underset{\sim}{b}$ and subtracting. For $\kappa^2 = -3\xi^2$ we obtain

$$a_1 = -1728\,(\theta - \psi)\tau\xi^8$$

$$b_1 = -6912\,\kappa(\theta - \psi)^2\tau^2\xi^6$$

$$c_1 = 27648(\theta - \psi)\tau^3\xi^8 \ . \qquad (7.27)$$

Evaluating $ca_1 - c_1a$, $bc_1 - b_1c$, and $ab_1 - a_1b$ by means of (7.24) and (7.27) and using $\kappa^2 = -3\xi^2$ once again, we verify that each of these expressions vanishes. We thus verify that $(\kappa^2 + 3\xi^2)^2$ is a factor of the general form of (7.15).

We need go no further. Either the repeated root $\kappa^2 = -111$ of (7.19) implies a factor $\kappa^2 + 3\xi^2$, or it does not. If it does not, then $\kappa^2 + 3\xi^2$ is not a common factor of the two conditions (7.15) and (7.19). If $\kappa^2 + 3\xi^2$ is a common factor then we have the possibility of satisfying the equations v

$$\kappa^2 + 3\xi^2 = 0 \ . \qquad (7.28)$$

This implies $\kappa = 0$ and is discounted by the analysis of Chapter 3. When (7.28) does not hold we may divide the factor out of the two conditions.

Case 2. The root $\kappa^2 = 37$

We suspect that this root corresponds to a factor $\kappa^2-\xi^2$. Setting $\kappa^2 = \xi^2$ in (6.15) we see that

$$a = 0 . \tag{7.29}$$

Setting $\kappa^2 = \xi^2$ in (6.15) we see that the condition (7.2) is now of the form

$$b \operatorname{div} \underset{\sim}{b} + c = 0 . \qquad (7.29)$$

The coefficient of $(\operatorname{div} \underset{\sim}{b})^3$ in the general form of (7.7) is easily verified to be $2\xi^2\kappa^2(3\xi^2-2\kappa^2)$. When $\kappa^2 = \xi^2$ this coefficient reduces to $2\xi^6$.

Thus when $\kappa^2=\xi^2$ the reduction of (7.7) to (7.12) by means of (7.2), amounts of simply to multiplying (7.29) by $2\xi^6$ div $\underset{\sim}{b}$. We thus obtain for this case

$$\begin{aligned} a_1 &= 2\xi^6 b , \\ b_1 &= 2\xi^6 c , \\ c_1 &= 0 . \end{aligned} \tag{7.30}$$

The eliminant of div $\underset{\sim}{b}$ now reduces to

$$(ca_1)^2 - a_1bb_1c = 0 \tag{7.31}$$

which is satisfied by (7.30).

We thus have the possibility

$$\kappa^2 - \xi^2 = 0 . \tag{7.32}$$

Taking the gradient of this condition twice with respect to $\underset{\sim}{b}$, using (2.11), (2.25) and (2.39), and cancelling κ, which cannot vanish, we get

$$\kappa \operatorname{div} \underset{\sim}{b} + 4\tau(\theta-\psi) = 0 \tag{7.33}$$

and

$$\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b} - (\operatorname{div} \underset{\sim}{b})^2 = -16\tau^2 . \tag{7.34}$$

The condition (4.14) reduces to

$$24\tau^2(\kappa + \operatorname{div} \underset{\sim}{n}) = \kappa(\operatorname{div} \underset{\sim}{b})^2 . \tag{7.35}$$

Substituting the values of $\theta-\psi$, $\frac{\delta}{\delta b} \operatorname{div} \underset{\sim}{b}$ and $(\kappa + \operatorname{div} \underset{\sim}{n})$ from (7.33), (7.34) and (7.35) into (2.29) we get

$$8\tau^2\kappa = 0 , \tag{7.36}$$

so that either τ or κ is zero. This is a contradiction. We may discount the possibility (7.32).

Write (7.15) and (7.19) in the forms

$$\begin{aligned} (\kappa^2 + 111)^2 (\kappa^2 - 57) P(\kappa^2) &= 0 \\ (\kappa^2 + 111)^2 (\kappa^2 - 57) Q(\kappa^2) &= 0 \end{aligned} \tag{7.37}$$

where $P(\kappa^2) = 0$ and $Q(\kappa^2) = 0$ do not have a common root in κ^2. The factors $\kappa^2 + 111$ and $\kappa^2 - 57$ represent the terms $\kappa^2 + 3\xi^2$ and $\kappa^2 - \xi^2$, and these expressions cannot vanish. We may divide out these factors and consider the equations

$$P(\kappa^2) = 0 \ ,$$

$$Q(\kappa^2) = 0 \ . \qquad (7.38)$$

Since equations (7.38) do not have a common root in κ^2 when $\theta-\psi$, λ and τ have the particular values (7.1), the eliminant of κ^2 from these equations does not vanish for these particular values. Thus the eliminant does not vanish identically. Setting the eliminant to zero, we have the relation $F(\theta-\psi,\lambda,\tau) = 0$. This proves Lemma 2.

8. THE FINAL PROOF

If the relation $F(\theta-\psi,\lambda,\tau) = 0$ is not degenerate, then since λ and τ are each constant on a vector-line of $\underset{\sim}{b}$, it follows that $\theta-\psi$ must be constant on a vector-line of $\underset{\sim}{b}$. It then follows from (2.25) that either τ or κ is zero. In the first case, by (2.3), the motion is irrotational, the second case was shown to be impossible in Chapter 3. If the relation is actually either of the forms $F(\theta-\psi,\lambda) = 0$, $F(\theta-\psi,\tau) = 0$, the same argument applies. If the relation is of the form $F(\lambda,\tau) = 0$ then it follows from (2.10) that

$$\frac{\delta\lambda}{\delta n} = 0 \ , \tag{8.1}$$

and hence by $(2.36)^2$ that

$$\frac{\delta^2\tau}{\delta n\delta s} = 0 \ . \tag{8.2}$$

From (8.2) and (2.28) either κ or $\frac{\delta\tau}{\delta s}$ must be zero. The first case has been shown to be impossible. If $\frac{\delta\tau}{\delta s}$ is zero it follows from (2.10) that τ is spatially constant. Hence by (2.3) Ω_b is spatially constant and the motion is a Trkalian motion. Evidently these arguments also dispose of the degeneracies $F(\theta-\psi) = 0$, $F(\tau) = 0$, $F(\lambda) = 0$. The only possibility beyond irrotational motion is the Trkalian motion. This proves the theorem.

9. CONCLUDING REMARKS

A further necessary condition, which was not used in the main proof may be obtained as follows. For steady stream lines the flow-wise or $\underset{\sim}{b}$-component of equation (1.4) yields

$$2\tau \frac{\partial v}{\partial t} = - \nu v \left[2(\theta-\psi) \frac{\delta\tau}{\delta s} - 2 \frac{\delta^2\tau}{\delta s^2} + 8\tau^3 \right] ,$$

so that

$$\frac{\partial}{\partial t} \log v = - \frac{\nu}{2\tau} \left[2(\theta-\psi) \frac{\delta\tau}{\delta s} - \frac{\delta^2\tau}{\delta s^2} + 8\tau^3 \right] . \tag{9.1}$$

Also, from (2.[illegible]), for steady stream lines

$$0 = \frac{\partial\theta}{\partial t} = - \frac{\partial}{\partial t} \left(\frac{\delta}{\delta s} \log v \right) = - \frac{\delta}{\delta s} \frac{\partial}{\partial t} \log v . \tag{9.2}$$

It follows from (9.1) and (9.2) that

$$\frac{\delta}{\delta s} \left[\frac{1}{\tau} \left[2(\theta-\psi) \frac{\delta\tau}{\delta s} - \frac{\delta^2\tau}{\delta s^2} + 8\tau^3 \right] \right] = 0 . \tag{9.3}$$

The expanded form of (9.3) is the condition in question. It is not helpful for the general proof.

The proof in Chapters 4 and 5 rested on the fact that the conditions were linear in the gradients $\frac{\delta\kappa}{\delta n}$, $\frac{\delta}{\delta n}$ div $\underset{\sim}{b}$, $\frac{\delta}{\delta s}(\theta-\psi)$, $\frac{\delta}{\delta n}(\theta-\psi)$, $\frac{\delta\kappa}{\delta s} + \theta\kappa$. These gradients could eventually be eliminated to give relations between κ, div $\underset{\sim}{b}$, $\theta-\psi$, and τ.

The condition (9.3) on the other hand involves products such as $\left(\frac{\delta\tau}{\delta s}\right)^2$, $\frac{\delta}{\delta s}(\theta-\psi)\,\frac{\delta\tau}{\delta s}$, $\frac{\delta\tau}{\delta s}\,\frac{\delta^2\tau}{\delta s^2}$. The expression (2.14) for $\frac{\delta\tau}{\delta s}$ involves the term $\theta + 2\psi$ and we get the formulae in terms of $\theta-\psi$ by using the expression (4.8) for ψ. This introduces the expression $\left(\frac{\delta\kappa}{\delta s} + \theta\kappa\right) + \frac{\delta}{\delta n}(\psi-\theta)$. We see that (9.3) will be quadratic in the gradients we require to eliminate. The resulting eliminants would be of correspondingly higher order than those used in the proof.

The particular case dealt with in Chapter 3 was a different matter. In this case the stream-line geometry was determined first so that a special simplified form of (9.3) could be used to verify the impossibility of the motion.

REFERENCES

1951 Bjørgum, O., "On Beltrami Vector Fields and Flows, Part I: A Comparative Study of Some Basic Types of Vector Fields. Universitet i Bergen Arbok 1951 Naturvitenskapelig rekke Nr 1.

1952 Bjørgum, O., and T. Godal, "On Beltrami Vector Fields and Flows, Part II: The Case when Ω is Constant in Space", Universitet i Bergen Arbok 1952, Naturvitenskapelig Nr 12.

1954 Ericksen, J. L., "Deformations Possible in Every Isotropic Incompressible Perfectly Elastic Body, Z. angew. Math. Phys. 5, 466-486.

1970 Marris, A. W., and C.-C. Wang, "Solenoidal Screw Fields of Constant Magnitude," Appendix. Arch. Rational Mech. Anal. 39, 227-244.

1979 Marris, A. W., "The Impossibility of Steady Screw Motions of a Navier-Stokes Fluid", Arch. Rational Mech. Anal. 70, 47-60.

Offprint from "Archive for Rational Mechanics and Analysis",
Volume 77, Number 1, 1981, pp. 95–102

Remarks on Plane Universal Navier-Stokes Motions

A. W. MARRIS

Communicated by C. TRUESDELL

Table of Contents

Introduction

Navier-Stokes motions are motions possible for an incompressible viscous fluid of uniform density and viscosity. Universal motions are motions for which the velocity field is independent of the viscosity, while the surface tractions required to produce the flow do depend on the viscosity. The instantaneous stream-line geometry is at once possible for all such viscous fluids but the stresses producing the motion, depending on the viscosity, vary from one fluid to the next. These motions are determined by the conditions

$$\operatorname{div} \boldsymbol{v} = 0, \tag{I.1}$$

$$\operatorname{curl} \boldsymbol{a} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \operatorname{curl} (\boldsymbol{\omega} \times \boldsymbol{v}) = \mathbf{0}, \tag{I.2}$$

and

$$\operatorname{curl} \operatorname{curl} \boldsymbol{\omega} = \mathbf{0}. \tag{I.3}$$

In the foregoing $\boldsymbol{v}$ is the velocity, $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}$ is the vorticity and $\boldsymbol{a}$ is the acceleration.

The case of plane flows with steady vorticity $\left(\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathbf{0}\right)$ was resolved by GÖRTLER & WIEGHARDT (1942 [1]). They showed that, beyond the rectilinear Poiseuille flow, the only possibilities are

Case 1. Motions with constant vorticity on which may be superposed a coplanar isochoric irrotational motion, which may be unsteady.

Case 2. Motions whose stream-lines are concentric circles, for which the velocity magnitude is

$$v = Ar \log r + Br + \frac{C}{r} \tag{I.4}$$

*where A, B and C are constants.**

In GÖRTLER & WIEGHARDT Case 2, one sees that the vorticity is

$$\boldsymbol{\omega} = [2A \log r + (A + 2B)]\, \boldsymbol{k}, \tag{I.5}$$

and

$$\operatorname{curl} \boldsymbol{\omega} = -\frac{2A}{r} \boldsymbol{e}_\theta, \tag{I.6}$$

where $\boldsymbol{k}$ is the unit vector perpenedicular to the plane of the motion; and $\boldsymbol{e}_\theta$ is the unit vector tangent to the stream-line.

It is apparent from (I.5) and (I.6) that the vector-lines of curl $\boldsymbol{\omega}$ are the stream-lines. Since the motion is steady, these are also material lines. Also the magnitude of $\boldsymbol{\omega}$ is a plane harmonic function which is constant on the vector-lines of curl $\boldsymbol{\omega}$.

Very little work seems to have been done on the question of unsteady plane flows possessing an acceleration potential (1954 [1], p. 93 footnote). However it appears that the properties outlined in the previous paragraph carry through for the unsteady motions. We prove the following *necessary* conditions for these motions.

The vector lines of curl $\boldsymbol{\omega}$ *are material lines on which the vorticity magnitude has a value which is temporally and spatially constant. These curves, themselves unsteady, are thus given by*

$$\omega(\boldsymbol{x}, t) = c. \tag{I.7}$$

The constant vorticity lines (I.7) *and their orthogonal trajectories*

$$\phi(\boldsymbol{x}, t) = d, \tag{I.8}$$

where curl $\boldsymbol{\omega}$ = grad ϕ, *are conjugate harmonic functions. Thus*

$$\omega + i\phi = F(z, t) \tag{I.9}$$

where $F(z, t)$ is a time dependent analytic function of the complex variable z.

In Chapter 3 we give an alternative proof of GÖRTLER & WIEGHARDT'S result for motions of steady vorticity.

---

* Unless the constant A is zero, the motion given in Case 2 leads to a many valued pressure if the stream-lines are complete circles.

1. Preliminary Formulae

We let $s(\boldsymbol{x}, t)$, and $\boldsymbol{n}(\boldsymbol{x}, t)$ be the time and position dependent unit vectors pointing along the instantaneous direction of the tangent and normal to the stream-lines at the spatial point $\boldsymbol{x}$. The constant unit vector perpendicular to the plane of the motion is denoted by $\boldsymbol{b}$. The velocity is then given by

$$\boldsymbol{v} = v(\boldsymbol{x}, t)\, \boldsymbol{s}(\boldsymbol{x}, t). \tag{1.1}$$

At any instant the spatial gradients of $\boldsymbol{s}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are given by

$$\begin{aligned} \operatorname{grad} \boldsymbol{s} &= \boldsymbol{s}\boldsymbol{n}\varkappa + \boldsymbol{n}\boldsymbol{n}\psi, \\ \operatorname{grad} \boldsymbol{n} &= -\boldsymbol{s}\boldsymbol{s}\varkappa - \boldsymbol{n}\boldsymbol{s}\psi, \\ \operatorname{grad} \boldsymbol{b} &= \boldsymbol{0}, \end{aligned} \tag{1.2}$$

and the scalars $\varkappa$ and ψ are identified as the curvatures of the vector-lines of $\boldsymbol{s}$ and $\boldsymbol{n}$ respectively. They depend on position $\boldsymbol{x}$ and time t.

We have

$$\operatorname{curl} \boldsymbol{s} = \varkappa \boldsymbol{b}, \quad \operatorname{curl} \boldsymbol{n} = \psi \boldsymbol{b}, \quad \operatorname{curl} \boldsymbol{b} = \boldsymbol{0} \tag{1.3}$$

and

$$\operatorname{div} \boldsymbol{s} = \psi, \quad \operatorname{div} \boldsymbol{n} = -\varkappa. \tag{1.4}$$

We also have

$$\frac{\partial \boldsymbol{s}}{\partial t} = \lambda \boldsymbol{n}, \quad \frac{\partial \boldsymbol{n}}{\partial t} = -\lambda \boldsymbol{s} \tag{1.5}$$

where λ is the angular velocity of rotation of $\boldsymbol{s}$ and $\boldsymbol{n}$ about the fixed direction $\boldsymbol{b}$ perpendicular to the plane of the motion.

We employ the notation $\frac{\delta F}{\delta s}$, $\frac{\delta F}{\delta n}$ to denote the components $\boldsymbol{s} \cdot \operatorname{grad} F$, $\boldsymbol{n} \cdot \operatorname{grad} F$, where F is a scalar field. Then

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\delta F}{\delta s}\right) &= \frac{\partial}{\partial t}(\boldsymbol{s} \cdot \operatorname{grad} F) = \frac{\partial \boldsymbol{s}}{\partial t} \cdot \operatorname{grad} F + \boldsymbol{s} \cdot \operatorname{grad} \frac{\partial F}{\partial t}, \\ &= \lambda \frac{\delta F}{\delta n} + \frac{\delta}{\delta s}\left(\frac{\partial F}{\partial t}\right), \quad \text{by } (1.5)^1. \end{aligned} \tag{1.6}$$

Similarly from $(1.5)^2$, we get

$$\frac{\partial}{\partial t}\left(\frac{\delta F}{\delta n}\right) = -\lambda \frac{\delta F}{\delta s} + \frac{\delta}{\delta n}\left(\frac{\partial F}{\partial t}\right). \tag{1.7}$$

Again the identity

$$\operatorname{curl} \operatorname{grad} F = \operatorname{grad} \times \left[\frac{\delta F}{\delta s} \boldsymbol{s} + \frac{\delta F}{\delta n} \boldsymbol{n}\right] = \boldsymbol{0}$$

yields the spatial commutation formula

$$\frac{\delta^2 F}{\delta n\, \delta s} - \frac{\delta^2 F}{\delta s\, \delta n} = \varkappa \frac{\delta F}{\delta s} + \psi \frac{\delta F}{\delta n}. \tag{1.8}$$

2. Analysis

The vorticity is given by (1.1) and (1.3) as

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v} = \omega \boldsymbol{b} = \left(\varkappa v - \frac{\delta v}{\delta n}\right) \boldsymbol{b}. \tag{2.1}$$

From (I.1) and $(1.4)^1$ we have

$$\frac{\delta v}{\delta s} + \psi v = 0. \tag{2.2}$$

We note that

$$\begin{aligned} \operatorname{curl} (\boldsymbol{\omega} \times \boldsymbol{v}) &= \operatorname{grad} (v\omega) \times \boldsymbol{n} + v\omega \operatorname{curl} \boldsymbol{n} \\ &= \left[\frac{\delta}{\delta s}(v\omega) + \psi v \omega\right] \boldsymbol{b}, \\ &= v \frac{\delta \omega}{\delta s} \boldsymbol{b}, \quad \text{by (2.2)}. \end{aligned} \tag{2.3}$$

The condition (I.2), necessary and sufficient for the motion to be circulation-preserving, now takes the form*

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + v \frac{\delta \omega}{\delta s} = \frac{\partial \omega}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \omega = 0. \tag{2.4}$$

The classical formula (2.4)** shows immediately that the curves $\omega(\boldsymbol{x}, t) = c$ are material lines. Our purpose is to show that in the case of unsteady motion these curves are the same as the family of vector-lines of curl $\boldsymbol{\omega}$.

By $(1.3)^3$ and (2.1)

$$\begin{aligned} \operatorname{curl} \boldsymbol{\omega} &= \operatorname{grad} \omega \times \boldsymbol{b} \\ &= \frac{\delta \omega}{\delta n} \boldsymbol{s} - \frac{\delta \omega}{\delta s} \boldsymbol{n}. \end{aligned} \tag{2.5}$$

We see from (2.4) and (2.5) that when curl $\boldsymbol{\omega} = \boldsymbol{0}$, the vorticity magnitude ω must be both spatially and temporally constant. The vorticity is a constant vector perpendicular to plane of the motion. It is straightforward to verify that

---

* We use the classical notation $\frac{D}{Dt}$ for material derivative.

** (1954 [1], p. 85, footnote.)

this motion allows the superposition of a co-planar isochoric irrotational motion, which may be unsteady. This is the first result given by GÖRTLER & WIEGHARDT (1942 [1]).

In the general case we see from (2.5) that at any instant the vector-lines of curl $\boldsymbol{\omega}$ are tangent to the family of curves

$$\omega(\boldsymbol{x}, t) = f(t). \tag{2.6}$$

We have to show that $f(t)$ must be a constant for a particular curve. We do this by first showing that the curves (2.6) are material lines and then using the fact that the curves $\omega(\boldsymbol{x}, t) = c$ are indeed material lines as required by (2.4).

According to the Helmholtz-Zorawski criterion (1960 [1], p. 341), a necessary and sufficient condition that the vector-lines of a vector field $\boldsymbol{c}(\boldsymbol{x}, t)$ are material lines is

$$\boldsymbol{c} \times \left[\frac{\partial \boldsymbol{c}}{\partial t} + \operatorname{curl}(\boldsymbol{c} \times \boldsymbol{v}) + \boldsymbol{v} \operatorname{div} \boldsymbol{c}\right] = \boldsymbol{0}. \tag{2.7}$$

We take $\boldsymbol{c}$ to be the vector curl $\boldsymbol{\omega}$. From (2.5), using (1.5), (1.6) and (1.7), we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \operatorname{curl} \boldsymbol{\omega} &= \frac{\partial}{\partial t}\left(\frac{\delta \omega}{\delta n}\right) \boldsymbol{s} + \frac{\delta \omega}{\delta n} \frac{\partial \boldsymbol{s}}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\delta \omega}{\delta s}\right) \boldsymbol{n} - \frac{\delta \omega}{\delta s} \frac{\partial \boldsymbol{n}}{\partial t} \\
&= \left(-\lambda \frac{\delta \omega}{\delta s} + \frac{\delta}{\delta n}\left(\frac{\partial \omega}{\partial t}\right)\right) \boldsymbol{s} + \lambda \boldsymbol{n} \frac{\delta \omega}{\delta n} \\
&\quad - \left(\lambda \frac{\delta \omega}{\delta n} + \frac{\delta}{\delta s}\left(\frac{\partial \omega}{\partial t}\right)\right) \boldsymbol{n} + \lambda \boldsymbol{s} \frac{\delta \omega}{\delta s} \\
&= \frac{\delta}{\delta n}\left(\frac{\partial \omega}{\partial t}\right) \boldsymbol{s} - \frac{\delta}{\delta s}\left(\frac{\partial \omega}{\partial t}\right) \boldsymbol{n}.
\end{aligned} \tag{2.8}$$

Again, by (2.5)

$$(\operatorname{curl} \boldsymbol{\omega}) \times \boldsymbol{v} = v \frac{\delta \omega}{\delta s} \boldsymbol{b},$$

so that by (1.2)

$$\operatorname{curl}\left[(\operatorname{curl} \boldsymbol{\omega}) \times \boldsymbol{v}\right] = \frac{\delta}{\delta n}\left(v \frac{\delta \omega}{\delta s}\right) \boldsymbol{s} - \frac{\delta}{\delta s}\left(v \frac{\delta \omega}{\delta s}\right) \boldsymbol{n}. \tag{2.9}$$

From (2.4), (2.8) and (2.9), we have, for $\boldsymbol{c} = \operatorname{curl} \boldsymbol{\omega}$,

$$\begin{aligned}
\frac{\partial \boldsymbol{c}}{\partial t} + \operatorname{curl}(\boldsymbol{c} \times \boldsymbol{v}) + \boldsymbol{v} \operatorname{div} \boldsymbol{c} &= \frac{\delta}{\delta n}\left(\frac{\partial \omega}{\partial t} + v \frac{\delta \omega}{\delta s}\right) \boldsymbol{s}, \\
&\quad - \frac{\delta}{\delta s}\left(\frac{\partial \omega}{\partial t} + v \frac{\delta \omega}{\delta s}\right) \boldsymbol{n} = \boldsymbol{0}.
\end{aligned} \tag{2.10}$$

The condition (2.10) taken with ZORAWSKI's criterion for flux conservation (1960 [1], p. 346) actually asserts that the flux of curl $\boldsymbol{\omega}$ across every material circuit remains constant in time. In any event (2.7) is satisfied, and we see that the vector-lines of curl $\boldsymbol{\omega}$, namely, the curves (2.6), must be material lines.

It follows that the material derivative of the function $\omega(\boldsymbol{x}, t) - f(t)$ must be zero; thus

$$\frac{\partial \omega}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \omega - \frac{D}{Dt} f(t) = 0. \tag{2.11}$$

Accordingly by (2.4) and (2.11), $f(t)$ must be constant on a given vector-line of curl $\boldsymbol{\omega}$. The vector lines of curl $\boldsymbol{\omega}$ are thus given by

$$\omega(\boldsymbol{x}, t) = c. \tag{I.7}$$

Finally, by (I.3), curl $\boldsymbol{\omega} = \operatorname{grad} \phi$, and the curves $\phi(\boldsymbol{x}, t) = d$ are the orthogonal trajectories of the curves (I.7). The functions ω and ϕ are plane harmonic functions satisfying the Cauchy-Riemann equations, and we have the representation (I.9).

3. Alternative for Proof the Case of Steady Vorticity

The original proof of GÖRTLER & WIEGHARDT for the case of steady vorticity (1942 [1]) employed the classical stream function for the two-dimensional velocity field.* We conclude by giving an alternative proof based on the present formalism.

When the vorticity is steady it follows from (2.4) and (2.5) that

$$\frac{\delta \omega}{\delta s} = 0 \tag{3.1}$$

and

$$\operatorname{curl} \boldsymbol{\omega} = \frac{\delta \omega}{\delta n} \boldsymbol{s}. \tag{3.2}$$

The vector-lines of curl $\boldsymbol{\omega}$ are now the stream-lines.

The condition (I.3) now gives

$$\frac{\delta^2 \omega}{\delta n^2} - \varkappa \frac{\delta \omega}{\delta n} = 0. \tag{3.3}$$

Again, since div curl $\boldsymbol{\omega} = 0$, we have by (2.2) and (3.2)

$$\frac{\delta^2 \omega}{\delta s \delta n} + \psi \frac{\delta \omega}{\delta n} = 0. \tag{3.4}$$

* They also invoke a theorem of HAMEL which delimits the steady plane motions of a viscous fluid which have the same stream-lines as an irrotational motion (1963 [1], p. 30, 1917 [1]). Professor W.-L. YIN constructed a proof on the lines of GÖRTLER & WIEGHARDT in which this dependence is avoided.

Since $\frac{\delta\omega}{\delta n}$ is non-vanishing we may write (3.3) and (3.4) in the respective forms

$$\varkappa = \frac{\delta h}{\delta n}, \tag{3.5}$$

$$\psi = -\frac{\delta h}{\delta s},$$

where

$$h = \log\left|\frac{\delta\omega}{\delta n}\right|. \tag{3.6}$$

From (3.5) and (1.8)

$$\frac{\delta\varkappa}{\delta s} + \frac{\delta\psi}{\delta n} = 0. \tag{3.7}$$

By (2.1) and (3.1) one has

$$\frac{\delta}{\delta s}\left(\varkappa v - \frac{\delta v}{\delta n}\right) = 0$$

so that

$$v\frac{\delta\varkappa}{\delta s} + \varkappa\frac{\delta v}{\delta s} - \frac{\delta^2 v}{\delta s\,\delta n} = 0. \tag{3.8}$$

Also by (1.8) and (2.2)

$$\begin{aligned}\frac{\delta^2 v}{\delta s\,\delta n} &= -\frac{\delta}{\delta n}(\psi v) - \varkappa\frac{\delta v}{\delta s} - \psi\frac{\delta v}{\delta n}\\ &= -v\frac{\delta\psi}{\delta n} - 2\psi\frac{\delta v}{\delta n} - \varkappa\frac{\delta v}{\delta s}.\end{aligned} \tag{3.9}$$

Thus by (3.8) and (3.9)

$$v\left(\frac{\delta\varkappa}{\delta s} + \frac{\delta\psi}{\delta n}\right) + 2\left(\varkappa\frac{\delta v}{\delta s} + \psi\frac{\delta v}{\delta n}\right) = 0, \tag{3.10}$$

By $(2.1)^2$, (2.2), (3.7) and (3.10) we find that

$$\psi\omega = 0 \tag{3.11}$$

so that, for rotational motion, $\psi = 0$, and the vector-lines of $\boldsymbol{n}$ are straight lines. By (3.7) $\frac{\delta\varkappa}{\delta s} = 0$, so the vector-lines of $\boldsymbol{s}$ are concentric circles. We write $\varkappa = \frac{1}{r}$. By (2.2) and (3.1), v and ω are functions of r only. Equation (3.3) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\delta\omega}{\delta r}\right) = 0,$$

where, by (2.1),

$$\omega = \frac{1}{r}\frac{\partial}{\partial r}(rv).$$

By direct integration we obtain the velocity distribution (I.4).

Acknowledgment. This work was supported by a grant to A. W. MARRIS from the National Science Foundation. This grant is gratefully acknowledged.

The author thanks Professor W.-L. YIN for his discussions and Professor J. L. ERICKSEN for reviewing the paper.

References

1917 [1] G. HAMEL, Spiralförmige Bewegungen zäher Flüssigkeiten. *Jahresbericht der Deutschen Mathematiker-Vereinigung XXV*, 34–60.

1942 [1] H. GÖRTLER & K. WIEGHARDT, Über eine gewisse Klasse von Strömungen zäher Flüssigkeiten und eine Kennzeichnung der Poiseuille-Strömung. *Mathematische Zeitschrift* **48**, 247–250.

1954 [1] C. TRUESDELL, *The Kinematics of Vorticity*, Indiana University Press, Bloomington.

1960 [1] C. TRUESDELL & R. TOUPIN, The Classical Field Theories. *Handbuch der Physik*, *III*/1, Berlin-Göttingen-Heidelberg: Springer.

1963 [1] R. BERKER, Intégration des équations du mouvement d'un fluide visqueux incompressible. *Handbuch der Physik*. *VIII*/2, Berlin-Göttingen-Heidelberg: Springer.

School of Engineering Science & Mechanics
Georgia Institute of Technology
Atlanta

(Received March 4, 1981)

Reprint from

ZAMP

Zeitschrift für angewandte Mathematik und Physik
Journal of Applied Mathematics and Physics
Journal de Mathématiques et de Physique appliquées

On circulation-preserving motions with lamellar vorticity

A. W. Marris

Vol. 33
Nr. 1
January 1982
pages 124–131

Birkhäuser Verlag
Basel · Boston · Stuttgart

Journal of Applied Mathematics and Physics (ZAMP)
Vol. 33, January 1982
0044-2275/82/001124-08 $ 3.10/0

On circulation-preserving motions with lamellar vorticity

By A. W. Marris, School of Engineering Science and Mechanics, Georgia Institute of Technology, Atlanta, Georgia, USA

Introduction

Isochoric circulation-preserving motions with steady vorticity are determined by the conditions

$$\operatorname{div} \boldsymbol{v} = 0\,, \tag{I.1}$$

and

$$\operatorname{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = 0\,, \tag{I.2}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ is the velocity at time t at the spatial point $\boldsymbol{x}$ and

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v} \tag{I.3}$$

is the vorticity. The vorticity $\boldsymbol{\omega}$ is independent of time in the spatial representation.

We prove the following proposition concerning these motions.

Theorem. *The only isochoric circulation-preserving motions of steady vorticity, for which*

1) *the vorticity is lamellar, so that*

$$\operatorname{curl} \boldsymbol{\omega} = 0\,, \tag{I.4}$$

and

2) *the vorticity magnitude bears a constant value on a particular vector-line of* $\boldsymbol{\omega}$, *so that*

$$\boldsymbol{\omega} \cdot \operatorname{grad} |\boldsymbol{\omega}| = 0\,, \tag{I.5}$$

are

1) *a plane motion of constant vorticity on which may be superposed a coplanar isochoric irrotational motion, which may be unsteady, together with a time dependent rigid translation perpendicular to the plane of the motion*

2) *the helical motion whose velocity has the physical components*

$$v_r = 0, \quad v_\theta = a\,r + \frac{f(t)}{r}\,, \quad v_z = b \log r + g\,(t)\,, \tag{I.6}$$

where a and b are arbitrary constants and $h(t)$ *and* $g(t)$ *are arbitrary functions of time.*

This result is a corollary of a theorem that was first given, with an inconclusive proof, by Hamel [1]. It was subsequently proved by the writer [2], [3]. Hamel's theorem is as follows. Let $\alpha(x)$ be a sufficiently smooth vector field in three-dimensional space, such that curl $\boldsymbol{\alpha}=0$, div $\boldsymbol{\alpha}=0$ and $\boldsymbol{\alpha}\cdot\mathrm{grad}|\boldsymbol{\alpha}|=0$, then the vector lines of $\boldsymbol{\alpha}$ must be parallel straight lines or circular helices mounted on concentric circular cylinders, so that

$$\boldsymbol{\alpha}=\frac{k_1}{r}\boldsymbol{e}_\theta+k_2\boldsymbol{e}_z\,, \tag{I.7}$$

where k_1, k_2 are constants, r is the distance from the axis and $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_z$ are unit vectors associated with cylindrical co-ordinates.

Since curl $\boldsymbol{\omega}$ is solenoidal we may take $\boldsymbol{\alpha}$ to be $\boldsymbol{\omega}$. Accordingly, when (I.4) and (I.5) hold, the vorticity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega}=\frac{k_1}{r}\boldsymbol{e}_\theta+k_2\boldsymbol{e}_z\,. \tag{I.8}$$

The present theorem is proved by a straight forward integration of (I.1), (I.2), (I.3) and (I.8).

The result allows one to delimit a certain class of universal motions of Navier-Stokes fluids of steady vorticity.

Motions of Navier-Stokes fluids for which the vorticity is steady are given by (I.1) and

$$\mathrm{curl}\,(\boldsymbol{\omega}\times\boldsymbol{v})=-\nu\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\omega}\,, \tag{I.9}$$

where ν is the kinematic viscosity. Universal motions are motions in which the velocity field is the same for all Navier-Stokes fluids. It is therefore independent of the viscosity. At the same time the stresses producing the motion may depend on the viscosity. These motions are determined by (I.1), (I.2) and the condition

$$\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\omega}=0\,. \tag{I.10}$$

Consider the subset of universal motions for which (I.10) is satisfied by the vanishing of curl $\boldsymbol{\omega}$. Even though the fluid is viscous the viscous stresses are inoperative. Suppose also that the vorticity magnitude is constant along a vortex-line, so that (I.5) holds. The present theorem requires that the only possible motions are the plane motions described and the motions given by (I.6).

More generally we may consider the universal motions of Navier-Stokes fluids of steady vorticity for which the magnitude of curl $\boldsymbol{\omega}$ bears a constant value on a vector-line of curl $\boldsymbol{\omega}$, thus

$$\mathrm{curl}\,\boldsymbol{\omega}\cdot\mathrm{grad}\,|\mathrm{curl}\,\boldsymbol{\omega}|=0\,. \tag{I.11}$$

In this case Hamel's theorem asserts that

$$\mathrm{curl}\,\boldsymbol{\omega}=\frac{k_1}{r}\boldsymbol{e}_\theta+k_2\boldsymbol{e}_z\,. \tag{I.12}$$

The vector field of curl $\boldsymbol{\omega}$ is established. The difficulty is the integration of (I.1), (I.2), (I.3) and (I.12). We may note that for universal motions for which the velocity field is complex-lamellar [4] and also for the class of universal motions given in [5], the condition (I.12) holds with $k_1=0$.

We proceed with the proof of the given theorem. The vorticity, given by (I.8), is a specialization of the vorticity of the circular helical motion of Strakhovitch ([6], p. 101) and one might expect the corresponding motion to be a specialization of this motion. However the Strakhovitch motion was obtained by postulating the form of the velocity field. When one takes the vorticity as the starting point one has the possibility of superposed irrotational motions. This was indeed the case in the analysis of helical vorticity fields given in [5]. For this reason we present the integration in detail.

Preliminaries

1. The case $k_1 = 0$

With $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v}$ given by (I.8), the velocity components v_r, v_θ, v_z are given by

$$\frac{\partial v_z}{r\,\partial\theta} - \frac{\partial v_\theta}{\partial z} = 0\,, \tag{1.1}$$

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = \frac{k_1}{r}\,, \tag{1.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_\theta) - \frac{\partial v_r}{r\,\partial\theta} = k_2\,, \tag{1.3}$$

where by (I.1),

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial v_\theta}{r\,\partial\theta} + \frac{\partial v_z}{\partial z} = 0\,. \tag{1.4}$$

We also have

$$\boldsymbol{\omega} \times \boldsymbol{v} = \left[\frac{k_1}{r}v_z - k_2 v_\theta\right]\boldsymbol{e}_r + k_2 v_r \boldsymbol{e}_\theta - \frac{k_1}{r}v_r \boldsymbol{e}_z\,. \tag{1.5}$$

The conditions (I.2) then yields the equations

$$\frac{k_1}{r^2}\frac{\partial v_r}{\partial\theta} + k_2\frac{\partial v_r}{\partial z} = 0\,, \tag{1.6}$$

$$\frac{\partial}{\partial z}\left(\frac{k_1}{r}v_z - k_2 v_\theta\right) + k_1\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) = 0\,, \tag{1.7}$$

$$\frac{k_2}{r}\frac{\partial}{\partial r}(r\,v_r) - \frac{\partial}{r\,\partial\theta}\left(\frac{k_1}{r}v_z - k_2 v_\theta\right) = 0\,. \tag{1.8}$$

Let us first suppose that $k_1 = 0$, $k_2 \neq 0$, so that the motion is one of constant vorticity

$$\boldsymbol{\omega} = k_2\,\boldsymbol{e}_z\,. \tag{1.9}$$

In this case by (1.6) and (1.7)

$$\frac{\partial v_r}{\partial z} = 0, \quad \frac{\partial v_\theta}{\partial z} = 0\,, \tag{1.10}$$

and by (1.4) and (1.8)

$$\frac{\partial v_z}{\partial z} = 0\,. \tag{1.11}$$

The conditions (1.1) and (1.2) now give

$$\frac{\partial v_z}{\partial\theta} = 0 \quad\text{and}\quad \frac{\partial v_z}{\partial r} = 0\,. \tag{1.12}$$

From (1.11) and (1.12)

$$v_z = f(t) \,. \tag{1.13}$$

The conditions (1.3) and (1.4) reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_\theta) - \frac{\partial v_r}{r\,\partial\theta} = k_2 \,, \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{\partial v_\theta}{r\,\partial\theta} = 0 \,,$$

and these equations taken with (1.10) and (1.11) determine an isochoric plane motion of constant vorticity on which an arbitrary plane isochoric irrotational motion may be superposed. Including the condition (1.13) we have the first motion stated in the theorem.

2. The case $k_1 \neq 0, k_2 \neq 0$

We consider the case when neither of the constants in (I.8) are zero. By (1.4) and (1.8)

$$\frac{k_1}{r^2}\frac{\partial v_z}{\partial\theta} + k_2\frac{\partial v_z}{\partial z} = 0 \,, \tag{2.1}$$

and by (1.1) and (2.1)

$$\frac{\partial}{\partial z}\left[\frac{k_1}{r}v_\theta + k_2\,v_z\right] = 0 \,, \tag{2.2}$$

and it follows that

$$\frac{k_1}{r}v_\theta + k_2\,v_z = \varphi(r, \theta, t) \,. \tag{2.3}$$

Again by (1.4) and (1.7)

$$\frac{k_1}{r}\left(\frac{\partial v_\theta}{r\,\partial\theta} + \frac{2v_r}{r}\right] + k_2\frac{\partial v_\theta}{\partial z} = 0 \,, \tag{2.4}$$

and by (1.1) and (2.4)

$$\frac{\partial}{\partial\theta}\left[\frac{k_1}{r}v_\theta + k_2\,v_z\right] = -2\frac{k_1\,v_r}{r} \,. \tag{2.5}$$

From (2.2) and (2.5) we have

$$\frac{\partial v_r}{\partial z} = 0 \,, \tag{2.6}$$

and then by (1.6)

$$\frac{\partial v_r}{\partial\theta} = 0 \,. \tag{2.7}$$

From (1.2) and (2.6) we obtain

$$\frac{\partial v_z}{\partial r} = -\frac{k_1}{r} \,. \tag{2.8}$$

Eliminating $\partial v_\theta/\partial z$ and $\partial v_z/\partial\theta$ from (1.1), (1.7) and (2.1) we have

$$\left[\frac{k_1^2 + k_2^2\,r^2}{k_1\,r}\right]\frac{\partial v_z}{\partial z} + k_1\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) = 0 \,, \tag{2.9}$$

and by (2.6) and (2.9),

$$\frac{\partial^2 v_z}{\partial z^2} = 0 . \tag{2.10}$$

From (2.10)

$$v_z = F(r, \theta, t) + G(r, \theta, t)\, z . \tag{2.11}$$

By (2.8) $\partial v_z/\partial r$ is not a function of z, thus

$$\frac{\partial G}{\partial r}(r, \theta, t) = 0$$

or

$$G(r, \theta, t) = \lambda(\theta, t) . \tag{2.12}$$

By (2.8), (2.11) and (2.12)

$$\frac{\partial F}{\partial r}(r, \theta, t) = -\frac{k_1}{r}$$

so that

$$F(r, \theta, t) = -k_1 \log r + \mu(\theta, t) . \tag{2.13}$$

By (2.11), (2.12) and (2.13)

$$v_z = -k_1 \log r + \mu(\theta, t) + \lambda(\theta, t)\, z . \tag{2.14}$$

By (2.1) and (2.14)

$$k_1 \left[\frac{\partial \mu}{\partial \theta}(\theta, t)\right] + k_1 \frac{\partial \lambda}{\partial \theta}(\theta, t)\, z = -k_2\, r^2\, \lambda(\theta, t) . \tag{2.15}$$

Since k_1 and k_2 are non-vanishing, and since the left hand side of (2.15) does not depend on r, we must have

$$\lambda(\theta, t) = 0 ,$$

and

$$\frac{\partial \mu(\theta, t)}{\partial \theta} = 0 ,$$

so that

$$\mu(\theta, t) = g(t) .$$

It now follows from (2.14) that

$$v_z = -k_1 \log r + g(t) . \tag{2.16}$$

From (1.1) and (2.16)

$$\frac{\partial v_\theta}{\partial z} = 0 , \tag{2.17}$$

and by (1.7), (2.16) and (2.17)

$$\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) = 0 . \tag{2.18}$$

By (2.6), (2.7) and (2.18)

$$v_r = \xi(t)\, r\,, \tag{2.19}$$

and by (1.3) and (2.7)

$$\frac{\partial}{\partial r}(r v_\theta) = k_2\, r\,. \tag{2.20}$$

Integrating (2.20), we have, by (2.17)

$$v_\theta = \frac{k_2 r}{2} + \frac{\chi(\theta, t)}{r}\,, \tag{2.21}$$

Substituting the expressions (2.19), (2.21) and (2.16) for v_r, v_θ and v_z respectively into (1.4), we obtain

$$2 r^2 \xi(t) = -\frac{\partial \chi(\theta, t)}{\partial \theta}\,.$$

It follows that

$$\xi(t) = 0$$

and

$$\chi(\theta, t) = h(t)\,.$$

Accordingly by (2.19) and (2.21) we have

$$v_r = 0, \qquad v_\theta = \frac{k_2 r}{2} + \frac{h(t)}{r}\,. \tag{2.22}$$

Equations (2.16) and (2.22) now establish the theorem.

3. The case $k_2 = 0, k_1 \neq 0$

When $k_2 = 0$ we deduce from (2.15) that

$$\lambda(\theta, t) = f(t)\,,$$

and

$$\mu(\theta, t) = g(t)\,,$$

so that

$$v_z = -k_1 \log r + g(t) + \mu(t)\, z\,. \tag{3.1}$$

The relation (2.17) still holds, so by (1.7) and (3.1)

$$\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right) + \frac{1}{r}\mu(t) = 0\,. \tag{3.2}$$

The relations (2.6) and (2.7) hold, so we integrate (3.2) to obtain

$$\frac{v_r}{r} = -\mu(t) \log r + s(t)\,. \tag{3.3}$$

By (1.3) and (2.7)

$$\frac{\partial}{\partial r}(r\,v_\theta) = 0$$

and so by (2.17)

$$v_\theta = \frac{\eta(\theta, t)}{r}. \tag{3.4}$$

By (2.5), when $k_2 = 0$, $k_1 \neq 0$,

$$\frac{\partial v_\theta}{\partial \theta} = -2k_1 v_r, \tag{3.5}$$

Substituting the expression (3.3) and (3.4) for v_r and v_θ into (3.5), we obtain

$$\frac{\partial}{\partial \theta}\eta(\theta, t) = 2k_1 r^2[\mu(t)\log r - s(t)]. \tag{3.6}$$

The condition (3.6) is impossible unless

$$\mu(t) = s(t) = 0, \tag{3.7}$$

and

$$\frac{\partial \eta(\theta, t)}{\partial \theta} = 0. \tag{3.8}$$

It follows from (3.1), (3.3), (3.4), (3.7) and (3.8) that

$$v_r = 0, \quad v_\theta = \frac{\eta(t)}{r}, \quad v_z = -k_1 \log r + g(t), \tag{3.9}$$

and the theorem is established once again.

Acknowledgement

This work was supported by a grant to A. W. Marris from the National Science Foundation. This grant is gratefully acknowledged.

The writer expresses his thanks to Professor J. L. Ericksen for reviewing the proofs in this paper.

References

[1] G. Hamel, *Potentialströmungen mit konstanter Geschwindigkeit.* Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 5–20 (1937).
[2] A. W. Marris and J. F. Shiau, *Hamel's theorem: the three polynomial integrals.* Rendiconti Circolo Matematico di Palermo, Serie 11-Tomo XXII, 185–216 (1973).
[3] A. W. Marris, *Hamel's theorem.* Arch. Rat. Mech. Analysis *51*, 85–105 (1973).
[4] A. W. Marris and W. F. Ames, *Addendum: On complex-lamellar motions.* Arch. Rat. Mech. Analysis *64*, 371–379 (1977).
[5] A. W. Marris and M. P. Stallybrass, *A class of fluid motions obtained by superposition.* Atti Accademia Scienze di Torino, 1-Classe di Scienze Fisiche, Matematiche e Naturali *113*, 53–60, (1979).
[6] R. Berker, *Intégration des équations du mouvement d'un fluide visqueux incompressible.* Handbuch d. Physik, Bd. VIII/2, Springer, Berlin 1963.

Summary

It is proved that the only isochoric circulation-preserving motions whose vorticity is steady, lamellar and bears a constant magnitude on a vortex-line are (1) plane motions of constant vorticity (on which certain unsteady potential motions may be superposed), and (2) a particular circular-helical motion.

Zusammenfassung

Es wird bewiesen, daß die einzigen isochoren Bewegungen mit Erhaltung der Zirkulation, deren Rotation zeitunabhängig lamellar und konstant entlang einer Wirbellinie ist, die folgenden sind: 1. Ebene Bewegung mit konstanter Rotation, der gewisse zeitabhängige Potentialbewegungen überlagert werden können; 2. eine besondere Kreis-Schrauben-Bewegung.

(Received: August 25, 1981.)

Offprint from "Archive for Rational Mechanics and Analysis",
Volume 79, Number 2, 1982, pp. 131–173

Two New Theorems on Ericksen's Problem

A. W. MARRIS

Communicated by C.-C. WANG

Table of Contents

Introduction

ERICKSEN (1954 [1]) attacked the general problem of determining the deformations that can be produced in every isotropic incompressible perfectly elastic body by the application of surface tractions when body forces are absent. Two cases proved intractable at that time, a case when the deformation tensor **c** had equal proper numbers and the case when its proper numbers were all constant. MARRIS & SHIAU (1970 [1]) showed there were no further solutions in the first category. The problem of constant proper numbers has remained unsolved.

We give a brief history of the researches on the case of constant proper numbers. FOSDICK (1966 [1]) noted that the known deformation

$$r = aR, \quad \theta = b\Theta, \quad z = cZ, \quad a^2bc = 1, \quad b \neq 1$$

represented a solution for the case of constant proper numbers. SINGH & PIPKIN (1965 [1]) gave the new solution

$$r = aR, \quad \theta = b \log R + c\Theta, \quad z = dZ, \quad a^2c\,d = 1 \tag{I.1}$$

and they noted that a special case of this deformation corresponding to $a^2c = 1$, $b^2 + c^2 = 1$, had been found by KLINGBEIL & SHIELD (1966 [2]). FOSDICK & SCHULER (1969 [1]) characterized all the universal deformations which were plane deformations (with uniform transverse stretch), and showed that, beyond homogeneous deformations, the above solution was the only plane deformation in the class of constant proper numbers. FOSDICK (1971 [1]) showed that there were no new solutions for the class of radially symmetric deformations. KAFADAR (1972 [1]) proved that (I.1) is the only possible solution in the case when any two of $\boldsymbol{c}$'s proper numbers are equal. For the remaining case of distinct and constant proper numbers, KAFADAR also proved that if the abnormality of the vector-field of any one of $\boldsymbol{c}$'s proper vectors vanishes, then no new solutions exist. Finally, the writer showed in (1975 [1]) that no new solutions exist when any two of the abnormalities of the proper vectors of $\boldsymbol{c}$ are constant.

It is seen that past work has been directed to showing that no new solutions exist for special classes of deformations. No attempt has been made to attack the problem from the other end, that is, to prove that if a new deformation exists it must be of a certain type. In the basic analysis of this paper and in Main Theorem 1 we take the latter point of view. On the other hand, Main Theorem 2 establishes the impossibility of a special class of deformations.

We prove the following theorems:

Main Theorem 1. *If there exists a new class of solutions, then it must be such that the curvatures and abnormalities of the fields of the unit proper vectors and the proper numbers are functionally related; thus*

$$F\left(\pi_{aa}, \pi_{ab}, \frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}, \frac{\sigma_3^2}{\sigma_1^2}\right) = 0 \tag{I.2}$$

*where F is a symmetric polynomial in its arguments and is homogeneous in the nine abnormalities and curvatures π_{aa}, π_{ab}. The form of F is such that it is invariant when the basis of proper vectors is transformed by reflection from a right-handed to a left-handed system, and for a ninety degree rotation about the direction determined by a proper vector.**

Main Theorem 2. *If the ratios of the abnormalities are constant, then there are no new solutions.*

---

* The proper numbers σ_a and abnormalities π_{aa} and curvatures π_{ab} referred to in (I.2) are defined in Chapter 1.

We give ten polynomial integrals effectively involving sixteen arguments. These are the ten variables indicated above and the six variables $u_1, u_2, u_3, \frac{\delta u_1}{\delta e_2}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$, defined in Chapter 1.* We prove Main Theorem 1 by showing that the above six variables can be eliminated from the ten integrals to yield at least one relation of the type (I.2). This is accomplished by substituting appropriate numerical values for the proper numbers, curvatures and abnormalities. We make use of the redundancy of integrals to simplify the calculation. This is discussed in Chapter 7.

In presenting this work we consider it important at the beginning to separate those conditions which are of geometrical origin from those arising from the equilibrium conditions. While at first sight the equilibrium conditions appear to give two vector equations, it turns out that these are not wholly independent of the geometrical conditions.

Chapters 1 and 2 deal with the geometrical conditions; the equilibrium conditions are introduced in Chapter 3. The ten basic integrals are developed in Chapters 4, 5 and 6. Main Theorem 1 is proved in Chapter 7 while Main Theorem 2 is proved in Chapters 8 and 9. In a final section, "Concluding Remarks", we indicate the various directions that have led to identities. Summarizing our futile attempts to obtain any further integrals of the same order as the original ten, it recounts the various checks that have been made on the basic conditions. We indicate the possible directions for obtaining higher order integrals and suggest that the problem may be suited to the new systems for treating symbolic mathematics on computers (1979 [1]).

Part 1. Proof of Main Theorem 1

1. Geometry of Isochoric Deformations with Constant Proper Numbers. Preliminaries

A deformation is defined by the invertible mapping $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$. Associated with the deformation are the deformation gradients $\boldsymbol{F}$ and $\boldsymbol{f}$ given by the double tensors**

$$\boldsymbol{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} = \sigma_1 \boldsymbol{e}_1 \boldsymbol{E}_1 + \sigma_2 \boldsymbol{e}_2 \boldsymbol{E}_2 + \sigma_3 \boldsymbol{e}_3 \boldsymbol{E}_3, \tag{1.1}$$

$$\boldsymbol{f} = \frac{d\boldsymbol{X}}{d\boldsymbol{x}} = \frac{1}{\sigma_1} \boldsymbol{E}_1 \boldsymbol{e}_1 + \frac{1}{\sigma_2} \boldsymbol{E}_2 \boldsymbol{e}_2 + \frac{1}{\sigma_3} \boldsymbol{E}_3 \boldsymbol{e}_3, \tag{1.2}$$

and the right and left Cauchy-Green tensors are $\boldsymbol{C}$ and $\boldsymbol{c}^{-1}$ where

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \sigma_1^2 \boldsymbol{E}_1 \boldsymbol{E}_1 + \sigma_2^2 \boldsymbol{E}_2 \boldsymbol{E}_2 + \sigma_3^2 \boldsymbol{E}_3 \boldsymbol{E}_3 \tag{1.3}$$

---

* See *Note added in proof* at the and of the paper.

** The symbolism $\boldsymbol{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}}$ indicates that $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$.

and

$$c = f^T f = \frac{1}{\sigma_1^2} e_1 e_1 + \frac{1}{\sigma_2^2} e_2 e_2 + \frac{1}{\sigma_3^2} e_3 e_3. \tag{1.4}$$

The ortho-normal bases E_A and e_a point along the proper vectors of C and c, and $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and their reciprocals are the corresponding proper numbers. They are considered to be constant and distinct. For an isochoric deformation we require that

$$\sigma_1 \sigma_2 \sigma_3 = 1. \tag{1.5}$$

The gradient of the vector field $e_c(x)$ referred to the basis e_a is written

$$\operatorname{grad} e_c = \gamma^b_{ac} e_a e_b, \quad \gamma^c_{ac} = 0, \quad \gamma^b_{ac} = -\gamma^c_{ab}. \tag{1.6}$$

The gradient of $E_c(X)$ is given similarly by

$$\operatorname{GRAD} E_c = \Gamma^B_{AC} e_A e_B, \quad \Gamma^C_{AC} = 0, \quad \Gamma^B_{AC} = -\Gamma^C_{AB}. \tag{1.7}$$

Rather than the gammas, we choose as the main vehicle for our analysis the functions π_{ab}, which are the components of curl e_a; thus

$$\pi_{ab} = e_b \cdot \operatorname{curl} e_a,$$

so that

$$\operatorname{curl} e_1 = \pi_{11} e_1 + \pi_{12} e_2 + \pi_{13} e_3, \text{ etc.}^{\star} \tag{1.8}$$

The functions $\pi_{11}, \pi_{22}, \pi_{33}$ are called abnormalities while the functions π_{ab}, $a \neq b$ are given the generic title of curvatures. A correponding set of pi functions is given by

$$\Pi_{AB} = E_B \cdot \operatorname{curl} E_A. \tag{1.9}$$

The functions π_{ab} and $\gamma^a_{bc} = -\gamma^c_{ba}$ are related as follows:

$$\begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} = \begin{bmatrix} \gamma^1_{32} - \gamma^1_{23} & \gamma^1_{13} & \gamma^2_{11} \\ \gamma^3_{22} & \gamma^2_{13} - \gamma^2_{31} & \gamma^2_{21} \\ \gamma^3_{32} & \gamma^1_{33} & \gamma^3_{21} - \gamma^3_{12} \end{bmatrix} \tag{1.10}$$

$$2\gamma^1_{32} = \pi_{11} + \pi_{22} - \pi_{33}, \quad 2\gamma^3_{21} = \pi_{33} + \pi_{11} - \pi_{22}, \quad 2\gamma^2_{13} = \pi_{22} + \pi_{33} - \pi_{11}.$$

Similar relations hold for the upper case variables.

The condition curl grad $F = \mathbf{0}$ applied to the scalar field F, yields, by (1.8) the basic commutation formulae**

$$\frac{\delta^2 F}{\delta e_3\, \delta e_2} - \frac{\delta^2 F}{\delta e_2\, \delta e_3} = \pi_{11} \frac{\delta F}{\delta e_1} + \pi_{21} \frac{\delta F}{\delta e_2} + \pi_{31} \frac{\delta F}{\delta e_3}, \text{ etc.} \tag{1.11}$$

---

* Throughout the work the appendage "*etc.*" will indicate that three equations, obtained by cyclic permutation of the subscripts, are implied.

** We use the symbol $\frac{\delta F}{\delta e_a}$ to denote the component $e_a \cdot \operatorname{grad} F$. Then $\frac{\delta^2 F}{\delta e_a\, \delta e_b} = e_a \cdot \operatorname{grad}(e_b \cdot \operatorname{grad} F)$, and so on.

We indicate two symmetry conditions which must be satisfied by all the general relations derived in the analysis. These conditions are thus extremely important as checks for the equations.

1. The reflection condition

If the unit vector $\boldsymbol{e}_1$ is changed to $-\boldsymbol{e}_1$, while $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are unaltered, the basis comprising the unit proper vectors of $\boldsymbol{c}$ and $\boldsymbol{c}^{-1}$ changes from a right-handed to a left-handed system. However, $\boldsymbol{c}$ and $\boldsymbol{c}^{-1}$ are unaltered. The operator curl changes to minus curl. From (1.18) we obtain the following transformation:

$$\begin{aligned}
&\boldsymbol{e}_1^* \to -\boldsymbol{e}_1, \quad \boldsymbol{e}_2^* \to \boldsymbol{e}_2, \quad \boldsymbol{e}_3^* \to \boldsymbol{e}_3, \quad \pi_{11}^* \to -\pi_{11},\\
&\pi_{22}^* \to -\pi_{22}, \quad \pi_{33}^* \to -\pi_{33}, \quad \pi_{23}^* \to -\pi_{23}, \quad \pi_{32}^* \to -\pi_{32},\\
&\pi_{12}^* \to \pi_{12}, \quad \pi_{21}^* \to \pi_{21}, \quad \pi_{13}^* \to \pi_{13}, \quad \pi_{31}^* \to \pi_{31}, \text{ etc.}
\end{aligned} \tag{1.12}$$

The general conditions derived must allow this transformation.

2. The rotation condition

The deformations tensors $\boldsymbol{c}$ and $\boldsymbol{c}^{-1}$ are unaltered if one rotates the axes ninety degrees about $\boldsymbol{e}_3$, for example, and interchanges σ_1^2 and σ_2^2. The general conditions must be invariant under the transformation

$$\begin{aligned}
&\boldsymbol{e}_2^* \to -\boldsymbol{e}_1, \quad \boldsymbol{e}_1^* \to \boldsymbol{e}_2, \quad \boldsymbol{e}_3^* \to \boldsymbol{e}_3, \quad \pi_{11}^* \to \pi_{22},\\
&\pi_{22}^* \to \pi_{11}, \quad \pi_{33}^* \to \pi_{33}, \quad \pi_{12}^* \to -\pi_{21}, \quad \pi_{21}^* \to -\pi_{12},\\
&\pi_{23}^* \to -\pi_{13}, \quad \pi_{32}^* \to -\pi_{31}, \quad \pi_{31}^* \to \pi_{32}, \quad \pi_{13}^* \to \pi_{23},\\
&\sigma_1^{*2} \to \sigma_2^2, \quad \sigma_2^{*2} \to \sigma_1^2, \quad \sigma_3^{*2} \to \sigma_3^2, \text{ etc.}
\end{aligned} \tag{1.13}$$

These relations are easily derivable from (1.8).

The deformation tensors $\boldsymbol{C}$ and $\boldsymbol{c}$ are metric tensors in Euclidean spaces. The Riemann curvature tensor calculated with $\boldsymbol{C}$ or $\boldsymbol{c}$ as metric must vanish. We obtain these conditions by a direct means first suggested by YIN.

It follows from (1.1) that the bases $\boldsymbol{e}_a$ and $\boldsymbol{E}_A$ are related through the deformation by*

$$\boldsymbol{e}_a \cdot d\boldsymbol{x} = \boldsymbol{e}_a \cdot \boldsymbol{F}\, d\boldsymbol{X} = \sigma_a \boldsymbol{E}_a \cdot d\boldsymbol{X}. \tag{1.14}$$

When (1.5) holds, for (1.14) to be integrable one must have

$$\Pi_{ab} = \frac{\pi_{ab}}{\sigma_a \sigma_b} \tag{1.15}$$

and

$$\frac{\delta \Pi_{ab}}{\delta E_c} = \sigma_c \frac{\delta \Pi_{ab}}{\delta e_c} = \frac{\sigma_c}{\sigma_a \sigma_b} \frac{\delta \pi_{ab}}{\delta e_c}. \tag{1.16}$$

* No sum is implied by the repeated indices in equations (1.14), (1.15), and (1.16).

Since the bases $\boldsymbol{e}_a$ and $\boldsymbol{E}_A$ are each embedded in Euclidean spaces, the curvature tensors based on the connections γ^b_{ac} and Γ^B_{AC} must vanish. Thus

$$r_{cbd}^{\cdot\cdot\cdot a} = \frac{\delta}{\delta e_c}\gamma^a_{bd} - \frac{\delta}{\delta e_b}\gamma^a_{cd} + \gamma^a_{ce}\gamma^e_{bd} - \gamma^a_{be}\gamma^e_{cd} + (\gamma^e_{bc} - \gamma^e_{cb})\gamma^a_{ed} = 0 \qquad (1.17)$$

and

$$R_{CBD}^{\cdot\cdot\cdot A} = \frac{\delta}{\delta E_c}\Gamma^A_{BD} - \frac{\delta}{\delta E_B}\Gamma^A_{CD} + \Gamma^A_{CE}\Gamma^E_{BD} - \Gamma^A_{BE}\Gamma^E_{CD} + (\Gamma^E_{BC} - \Gamma^E_{CB})\Gamma^A_{ED} = 0. \qquad (1.18)$$

The conditions (1.17) and (1.18) may be expressed in terms of π_{ab} and Π_{AB} by (1.10). The condition involving Π_{AB} and $\dfrac{\delta \Pi_{AB}}{\delta E_c}$ is then transformed through (1.15) and (1.16) into a corresponding condition involving $\pi_{ab}, \dfrac{\delta\pi_{ab}}{\delta e_c}$ and $\sigma_1^2, \sigma_2^2, \sigma_3^2$. The method is presented fully in (1975 [1], p. 117, 118).*

There are nine independent conditions represented by each of (1.17) and (1.18). These occur in three sets of cyclic conditions, a specimen of each set being given by

$$r_{232}^{\cdot\cdot\cdot 3} = 0, \quad r_{232}^{\cdot\cdot\cdot 1} = 0, \quad r_{312}^{\cdot\cdot\cdot 1} = 0, \qquad (1.19)$$

$$R_{232}^{\cdot\cdot\cdot 3} = 0, \quad R_{232}^{\cdot\cdot\cdot 1} = 0, \quad R_{312}^{\cdot\cdot\cdot 1} = 0. \qquad (1.20)$$

We write

$$\alpha_{rs} \overset{\text{def}}{=} 1 - \frac{\sigma_r^2}{\sigma_s^2} \neq 0, \qquad (1.21)$$

and note the identities

$$\frac{\alpha_{12}\,\alpha_{23}\,\alpha_{31}}{\alpha_{21}\,\alpha_{32}\,\alpha_{13}} = -1, \qquad (1.22)$$

$$\frac{\alpha_{13}}{\alpha_{12}} \equiv 1 - \frac{\alpha_{23}}{\alpha_{21}}, \quad \text{etc.} \qquad (1.23)$$

From (1975 [1], equations (2.5) and (2.6)) we have, corresponding to $r_{232}^{\cdot\cdot\cdot 3} = 0$ and $R_{232}^{\cdot\cdot\cdot 3} = 0$,

$$\frac{\delta}{\delta e_3}\pi_{21} = \pi_{21}^2 - \frac{\alpha_{12}}{\alpha_{32}}\pi_{32}\pi_{23} + \xi_{321}, \quad \text{etc.,} \qquad (1.24)$$

---

* In (1975 [1]) the curvatures were immediately expressed as gradient functions in accordance with the equilibrium conditions. This obscures the purely geometrical origin of many of the relations. Also in (1975 [1]) the condition (1.33) was not derived as a geometrical condition from (1.17) and (1.18). It happens, as will be seen, that this same condition also follows from the equilibrium conditions. In (1975 [1]) it was obtained as a consequence of the equilibrium conditions. Here we indicate the geometrical origin of the conditions first to exhibit the geometrical problem as a separate problem and then to indicate the essential weakness of the equilibrium conditions.

where

$$\xi_{321} = -\frac{\pi_{11}\pi_{22}}{2} + \frac{\alpha_{12}}{\alpha_{32}}\frac{\pi_{22}\pi_{33}}{2} + \frac{3}{4}\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}^2 - \frac{1}{4}\left[\frac{\alpha_{32}}{\alpha_{31}} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{31}\alpha_{23}}\right]\pi_{22}^2$$
$$- \frac{1}{4}\frac{\alpha_{13}}{\alpha_{32}}\pi_{33}^2, \quad \text{etc.} \tag{1.25}$$

and

$$\frac{\delta}{\delta e_2}\pi_{31} = -\pi_{31}^2 + \frac{\alpha_{13}}{\alpha_{23}}\pi_{32}\pi_{23} + \eta_{231}, \quad \text{etc.,} \tag{1.26}$$

where*

$$\eta_{231} = \frac{\pi_{11}\pi_{33}}{2} - \frac{\alpha_{13}}{\alpha_{23}}\frac{\pi_{22}\pi_{33}}{2} - \frac{3}{4}\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2$$
$$+ \frac{1}{4}\frac{\alpha_{13}}{\alpha_{12}}\left[\frac{\alpha_{13}}{\alpha_{23}} - \frac{\alpha_{32}}{\alpha_{31}}\right]\pi_{33}^2 + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{23}}\pi_{22}^2, \quad \text{etc.} \tag{1.27}$$

The conditions (1.26), (1.27) are dual to (1.24), (1.25) in that one set may be obtained from the other by applying the rotation condition (1.13).

From (1975 [1]), equation (2.13), we have, corresponding to $r_{232}^{\cdots 1} = 0$ and $R_{232}^{\cdots 1} = 0$,

$$\frac{\delta}{\delta e_2}\pi_{11} - 2\pi_{13}\pi_{11} - \frac{\alpha_{23}}{\alpha_{21}}\left(\frac{\delta}{\delta e_2}\pi_{33} + 2\pi_{31}\pi_{33}\right) = 0, \quad \text{etc.} \tag{1.28}$$

The set of three equations (1.28) transforms into itself under the rotation condition (1.13).

We now introduce

$$u_1 \overset{\text{def}}{=} \frac{\delta}{\delta e_2}\pi_{33} + 2\pi_{31}\pi_{33}, \quad \text{etc.,} \tag{1.29}$$

so that by (1.28)

$$\frac{\delta}{\delta e_2}\pi_{11} - 2\pi_{13}\pi_{11} = \frac{\alpha_{23}}{\alpha_{21}}u_1, \quad \text{etc.} \tag{1.30}$$

We note that under the transformation (1.13), the variables u_a transform as follows:

$$u_1^* \to -\frac{\alpha_{12}}{\alpha_{13}}u_3, \quad u_2^* \to \frac{\alpha_{31}}{\alpha_{32}}u_2, \quad u_3^* \to \frac{\alpha_{23}}{\alpha_{21}}u_1. \tag{1.31}$$

---

* To reconcile the forms (1.25) and (1.27) to (1975 [1], equations (2.5) and (2.6)), one may readily verify that

$$\frac{1}{\alpha_{32}}\left(1 - \frac{\sigma_1^2\sigma_3^2}{\sigma_2^4}\right) = \frac{\alpha_{32}}{\alpha_{31}} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{31}\alpha_{23}}$$

and

$$\frac{1}{\alpha_{23}}\left(1 - \frac{\sigma_1^2\sigma_2^2}{\sigma_3^4}\right) = \frac{\alpha_{13}}{\alpha_{12}}\left(\frac{\alpha_{13}}{\alpha_{23}} - \frac{\alpha_{32}}{\alpha_{31}}\right).$$

The second condition obtained from $r_{23\dot{2}}^{\cdot\cdot\cdot 1} = 0$ and $R_{23\dot{2}}^{\cdot\cdot\cdot 1} = 0$, obtained from (1975 [1], equations (2.14) and (1.30)), is

$$\frac{\delta}{\delta e_3}\pi_{23} = -\pi_{22}\pi_{13} + \pi_{21}(\pi_{23} + \pi_{32}) + \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}u_1, \quad \text{etc.} \tag{1.32}$$

Transforming (1.32) by (1.13) and (1.31), and using (1.22), we obtain the dual set

$$\frac{\delta}{\delta e_3}\pi_{13} = \pi_{11}\pi_{23} - \pi_{12}(\pi_{13} + \pi_{31}) - \frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}u_3, \quad \text{etc.} \tag{1.33}$$

We have the three conditions $\operatorname{div}\operatorname{curl} \boldsymbol{e}_a = 0$, so that, by (1.8),

$$\frac{\delta\pi_{11}}{\delta e_1} + \pi_{11}\operatorname{div}\boldsymbol{e}_1 + \frac{\delta}{\delta e_2}\pi_{12} + \pi_{12}\operatorname{div}\boldsymbol{e}_2 + \frac{\delta}{\delta e_3}\pi_{13} + \pi_{13}\operatorname{div}\boldsymbol{e}_3 = 0, \quad \text{etc.} \tag{1.34}$$

and from (1.6) and (1.10)

$$\operatorname{div}\boldsymbol{e}_1 = \pi_{23} - \pi_{32}, \quad \text{etc.} \tag{1.35}$$

We verify that (1.33) is indeed a geometrical condition corresponding to $r_{31\dot{2}}^{\cdot\cdot\cdot 1} = 0$. From (1.7), (1.10) and (1.17) we have for $r_{31\dot{2}}^{\cdot\cdot\cdot 1} = 0$

$$\begin{aligned} &-\frac{\delta}{\delta e_3}\pi_{13} - \frac{1}{2}\frac{\delta}{\delta e_1}(\pi_{11} + \pi_{22} - \pi_{33}) - \pi_{32}(\pi_{33} - \pi_{11}) \\ &-\pi_{12}(\pi_{31} + \pi_{13}) - \pi_{22}\pi_{23} = 0. \end{aligned} \tag{1.36}$$

Substituting for $\frac{\delta}{\delta e_1}\pi_{11}$, $\frac{\delta}{\delta e_1}\pi_{22}$, $\frac{\delta}{\delta e_1}\pi_{33}$ from (1.34), (1.29) and (1.30) respectively, using (1.32) for $\frac{\delta}{\delta e_2}\pi_{12}$ and using the expressions (1.35), we regain the condition (1.33).*

2. Geometry of Isochoric Deformations with Constant Proper Numbers. Three Small Theorems

Eliminating u_3 from the sets (1.32) and (1.33), we obtain

$$\frac{\delta}{\delta e_2}\pi_{12} + \frac{\delta}{\delta e_3}\pi_{13} + \pi_{12}\pi_{31} - \pi_{13}\pi_{21} - \pi_{11}(\pi_{23} - \pi_{32}) = 0, \quad \text{etc.} \tag{2.1}$$

From (1.34) and (2.1) by using (1.35), we get

$$\frac{\delta\pi_{11}}{\delta e_1} + 2\pi_{11}\operatorname{div}e_1 = 0, \quad \text{etc.,} \tag{2.2}$$

* The nine independent conditions implied by (1.17) include the three conditions (1.34). The conditions (1.5), (1.15) and (1.16) mean that condition DIV CURL $E_A = 0$, computed for the images of the proper numbers before deformation, implies $\operatorname{div}\operatorname{curl}\boldsymbol{e}_a = 0$. We thus get fifteen rather than eighteen independent compatibility conditions for the deformation. These conditions are the sets (1.24), (1.26), (1.28), (1.34) and the conditions obtained by eliminating u_1, u_2, and u_3 from the sets (1.32) and (1.33).

or

$$\operatorname{div}\left(\pi_{11}^{\frac{1}{2}}\, e_1\right) = 0, \quad \text{etc.}$$

One has

Theorem 2.1. *The vector fields of the unit proper vectors* $\mathbf{e}_a$ *are complex-solenoidal fields:*

$$e_a = \frac{1}{\pi_{11}^{\frac{1}{2}}} \operatorname{curl} \boldsymbol{p}_a.$$

By (1.15) and (1.16) an analogous theorem holds for the unit proper vectors $\boldsymbol{E}_A$.

We now note that

$$\operatorname{div}(e_1 e_1) = \operatorname{div} e_1 e_1 + \pi_{13} e_2 - \pi_{12} e_3, \quad \text{etc.} \tag{2.3}$$

and that the three conditions (2.1) express

$$e_1 \cdot \operatorname{curl} \operatorname{div}(e_1 e_1) = 0, \quad \text{etc.} \tag{2.4}$$

We have

Theorem 2.2. *The condition* (2.4) *is a geometrical result for isochoric deformations with constant distinct proper numbers.*

The significance of Theorem 2.2 is that it tells us that the equilibrium conditions to be considered in the next section give only three cyclic scalar conditions, rather than the six they appear to do at first sight. It emphasizes the ultimate weakness of the equilibrium conditions.

From (1.29), (1.30) and (2.2)[1] we have for non-vanishing abnormalities

$$\begin{aligned} &\frac{1}{2}\left(\frac{u_3}{\pi_{22}} + \frac{\alpha_{12}}{\alpha_{13}}\frac{u_3}{\pi_{33}}\right) - 2 \operatorname{div} e_1 \\ &\quad = \frac{\delta}{\delta e_1} \log \pi_{22}^{\frac{1}{2}} + \pi_{23} + \frac{\delta}{\delta e_1} \log \pi_{33}^{\frac{1}{2}} - \pi_{32} + \frac{\delta}{\delta e_1} \log \pi_{11}^{\frac{1}{2}} - \operatorname{div} e_1. \end{aligned} \tag{2.5}$$

From (1.35) and (2.5) there follows:

Theorem 2.3. *For isochoric deformations with constant distinct proper numbers, and for which the abnormalities of the vector fields of* $\mathbf{e}_a$ *do not vanish, one has*

$$\begin{aligned} &\left[-4 \operatorname{div} e_1 + \frac{\lambda_1 u_3}{\pi_{22}\pi_{33}}\right] e_1 + \left[-4 \operatorname{div} e_2 + \frac{\lambda_2 u_1}{\pi_{33}\pi_{11}}\right] e_2 \\ &\quad + \left[-4 \operatorname{div} e_3 + \frac{\lambda_3 u_2}{\pi_{11}\pi_{22}}\right] e_3 = \operatorname{grad} \log |\pi_{11}\, \pi_{22}\, \pi_{33}|, \end{aligned} \tag{2.6}$$

where

$$\lambda_1 \overset{\text{def}}{=} \left(\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}}\, \pi_{22}\right), \quad \text{etc.} \tag{2.7}$$

3. Introduction of the Equilibrium Conditions

When the proper numbers σ_1^2, σ_2^2, σ_3^2 of c^{-1} are constant, the equilibrium conditions for the incompressible perfectly elastic material reduce to

$$\operatorname{curl}\operatorname{div} \boldsymbol{c} = \mathbf{0}$$

and

$$\operatorname{curl}\operatorname{div} \boldsymbol{c}^{-1} = \mathbf{0}. \tag{3.1}$$

For distinct proper numbers these conditions require that

$$\operatorname{curl}\operatorname{div}(\boldsymbol{e}_1\boldsymbol{e}_1) = \mathbf{0}, \quad \operatorname{curl}\operatorname{div}(\boldsymbol{e}_2\boldsymbol{e}_2) = \mathbf{0}, \quad \operatorname{curl}\operatorname{div}(\boldsymbol{e}_3\boldsymbol{e}_3) = \mathbf{0}, \tag{3.2}$$

where

$$\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3 = \boldsymbol{I}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$(\operatorname{div} \boldsymbol{e}_1)\, \boldsymbol{e}_1 + \pi_{13}\boldsymbol{e}_2 - \pi_{12}\boldsymbol{e}_3 = \operatorname{grad} \theta_1, \quad \text{etc.} \tag{3.4}$$

where

$$\theta_1 + \theta_2 + \theta_3 = \text{constant}.$$

The condition $(3.4)^2$ of course follows from (1.35).

Taking the curl of $(3.4)^1$, we obtain two independent conditions

$$\frac{\delta}{\delta e_1}\pi_{13} = \frac{\delta}{\delta e_2}\operatorname{div} \boldsymbol{e}_1 - \pi_{13}(2\pi_{23} - \pi_{32}) + \pi_{12}\pi_{33}, \quad \text{etc.},$$

$$\frac{\delta}{\delta e_1}\pi_{12} = -\frac{\delta}{\delta e_3}\operatorname{div} \boldsymbol{e}_1 - \pi_{12}(\pi_{23} - 2\pi_{32}) - \pi_{13}\pi_{22}, \quad \text{etc.} \tag{3.5}$$

Equations (3.5) are dual in that they can be derived from each other through the rotation condition (1.13).

It is important to note that the self-dual set of conditions (2.1), which are consequences of the geometrical conditions only, can be obtained from the conditions (3.5). Therefore the equilibrium conditions and the geometrical conditions are not wholly independent. This is evident also from equations (2.4) and (3.2).

We now introduce the subsidiary variables

$$x_1 \overset{\text{def}}{=} \frac{\delta}{\delta e_2}\operatorname{div} \boldsymbol{e}_3 + \pi_{31}\operatorname{div} \boldsymbol{e}_3, \quad \text{etc.},$$

$$y_1 \overset{\text{def}}{=} \frac{\delta}{\delta e_3}\operatorname{div} \boldsymbol{e}_2 - \pi_{21}\operatorname{div} \boldsymbol{e}_2, \quad \text{etc.} \tag{3.6}$$

Under the transformation (1.13) these variables x_a and y_a transform as follows:

$$x_1^* \to -y_2, \quad y_1^* \to -x_2, \quad x_2^* \to y_1, \quad y_2^* \to x_1,$$
$$x_3^* \to -y_3, \quad y_3^* \to -x_3. \tag{3.7}$$

From (3.5) we obtain the dual sets of conditions

$$\frac{\delta}{\delta e_1}\pi_{13} = y_3 - \pi_{13}\pi_{23} + \pi_{12}\pi_{33}, \quad \text{etc.}, \tag{3.8}$$

$$\frac{\delta}{\delta e_1}\pi_{12} = -x_2 + \pi_{12}\pi_{32} - \pi_{13}\pi_{22}, \quad \text{etc.} \tag{3.9}$$

We remember that (3.8) and (3.9) do imply the use of the equilibrium condition.

From (1.35) and (3.8)[1]

$$\frac{\delta}{\delta e_2}\pi_{21} = \frac{\delta}{\delta e_2}\pi_{12} - \frac{\delta}{\delta e_2}\operatorname{div} \boldsymbol{e}_3 = y_1 - \pi_{21}\pi_{31} + \pi_{23}\pi_{11},$$

whence, by (3.6) and (1.35),

$$\frac{\delta}{\delta e_2}\pi_{12} = x_1 + y_1 - \pi_{12}\pi_{31} + \pi_{23}\pi_{11} \tag{3.10}$$

so that by (1.32) we obtain the self-dual set of conditions

$$x_1 + y_1 + \pi_{11}(\pi_{23} + \pi_{32}) - \pi_{12}(\pi_{31} + \pi_{13}) - \pi_{13}\pi_{21} - \frac{\alpha_{32}}{2\alpha_{31}}u_3 = 0. \tag{3.11}$$

From (1.29), (1.30) and (3.4) we obtain

$$\frac{u_1}{2\pi_{33}} = \frac{\delta}{\delta e_2}\left(\log \pi_{33}^{\frac{1}{2}} - \theta_3\right), \quad \text{etc.}, \tag{3.12}$$

$$\frac{1}{2}\frac{\alpha_{12}}{\alpha_{13}}\frac{u_3}{\pi_{33}} = \frac{\delta}{\delta e_1}\left(\log \pi_{33}^{\frac{1}{2}} - \theta_3\right), \quad \text{etc.}, \tag{3.13}$$

while by (2.2)[1] and (3.4)

$$\frac{\delta}{\delta e_1}\left(\log \pi_{11}^{\frac{1}{2}} - \theta_1\right) = -2\operatorname{div} \boldsymbol{e}_1, \quad \text{etc.} \tag{3.14}$$

From (1.11) one has

$$\left[\frac{\delta^2}{\delta e_2\,\delta e_1} - \frac{\delta^2}{\delta e_1\,\delta e_2}\right]\left[\log \pi_{11}^{\frac{1}{2}} - \theta_1\right]$$
$$= \left[\pi_{13}\frac{\delta}{\delta e_1} + \pi_{23}\frac{\delta}{\delta e_2} + \pi_{33}\frac{\delta}{\delta e_3}\right]\left[\log \pi_{11}^{\frac{1}{2}} - \theta_1\right]$$

so that from (2.2), (3.12), (3.13) and (3.14),

$$-\frac{\delta}{\delta e_2}(2\operatorname{div} \boldsymbol{e}_1) - \frac{\delta}{\delta e_1}\left(\frac{\alpha_{23}}{\alpha_{21}}\frac{u_1}{2\pi_{11}}\right) = -2\pi_{13}\operatorname{div} \boldsymbol{e}_1 + \pi_{23}\frac{\alpha_{23}}{\alpha_{21}}\frac{u_1}{2\pi_{11}} + \pi_{33}\frac{u_2}{2\pi_{11}},$$

which, using (1.35) and (3.6)[2], reduces to

$$\frac{\alpha_{23}}{\alpha_{21}}\frac{\delta u_1}{\delta e_1} = -4\pi_{11}y_3 - (3\pi_{23} - 2\pi_{32})\frac{\alpha_{23}}{\alpha_{21}}u_1 - \pi_{33}u_2, \quad \text{etc.} \tag{3.15}$$

Again from

$$\left[\frac{\delta^2}{\delta e_3\,\delta e_2} - \frac{\delta^2}{\delta e_2\,\delta e_3}\right]\left[\log \pi_{33}^{\frac{1}{2}} - \theta_3\right]$$

$$= \left[\pi_{11}\frac{\delta}{\delta e_1} + \pi_{21}\frac{\delta}{\delta e_2} + \pi_{31}\frac{\delta}{\delta e_3}\right]\left[\log \pi_{33}^{\frac{1}{2}} - \theta_3\right]$$

and (1.35), (2.2), (3.12), (3.13), (3.14) and (3.6)[1], one obtains

$$\frac{\delta u_1}{\delta e_3} = -4\pi_{33}x_1 + (3\pi_{21} - 2\pi_{12})\,u_1 + \pi_{11}\frac{\alpha_{12}}{\alpha_{13}}u_3, \quad \text{etc.} \tag{3.16}$$

The conditions (3.16) may otherwise be obtained from (3.14) by applying the rotation transformations (1.13), (1.31) and (3.7).

Finally, treating in a similar manner the commutation

$$\left[\frac{\delta^2}{\delta e_3\,\delta e_2} - \frac{\delta^2}{\delta e_2\,\delta e_3}\right]\left[\log \pi_{11}^{\frac{1}{2}} - \theta_1\right]$$

$$= \left[\pi_{11}\frac{\delta}{\delta e_1} + \pi_{21}\frac{\delta}{\delta e_2} + \pi_{31}\frac{\delta}{\delta e_2}\right]\left[\log \pi_{11}^{\frac{1}{2}} - \theta_1\right],$$

one obtains the set

$$\frac{\alpha_{23}}{\alpha_{21}}\frac{\delta u_1}{\delta e_3} - \frac{\delta u_2}{\delta e_2} = -4\pi_{11}^2\,\mathrm{div}\,\mathbf{e}_1 + (\pi_{21} - 2\pi_{12})\frac{\alpha_{23}}{\alpha_{21}}u_1 + (\pi_{31} - 2\pi_{13})\,u_2, \quad \text{etc.} \tag{3.17}$$

Eliminating the gradients $\frac{\delta u_1}{\delta e_3}$ and $\frac{\delta u_2}{\delta e_2}$ from (3.17) by means of (3.15) and (3.16), and using (1.22), we get the integrals

$$\pi_{21}\frac{\alpha_{23}}{\alpha_{21}}u_1 + \pi_{31}u_2 - 2\frac{\alpha_{23}}{\alpha_{21}}\pi_{33}x_1 + 2\frac{\alpha_{32}}{\alpha_{31}}\pi_{22}y_1 + 2\pi_{11}^2\mathrm{div}\,\mathbf{e}_1 = 0, \quad \text{etc.}^{\star} \tag{3.18}$$

We may now solve (3.11) and (3.18) to give the x_α and y_α in terms of u_α. One obtains

$$\mu_1 x_1 = \frac{1}{2}\left(\frac{\alpha_{23}}{\alpha_{21}}\pi_{21}u_1 + \pi_{31}u_2 + \frac{\alpha_{32}\alpha_{32}}{\alpha_{31}\alpha_{31}}\pi_{22}u_3\right) - \frac{\alpha_{32}}{\alpha_{31}}\pi_{22}f_1 + \pi_{11}^2\,\mathrm{div}\,\mathbf{e}_1,$$

$$\mu_1 y_1 = -\frac{1}{2}\left(\frac{\alpha_{23}}{\alpha_{21}}\pi_{21}u_1 + \pi_{31}u_2 - \frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{31}}\pi_{33}u_3\right) - \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}f_1 - \pi_{11}^2\,\mathrm{div}\,\mathbf{e}_1, \quad \text{etc.} \tag{3.19}$$

⋆ The conditions (3.18) are the statement that the curl of the left-hand side (2.6) vanishes. In the subsequent analysis we shall take directional derivatives of (3.18). We must avoid the identity represented by div curl $= 0$.

where

$$\mu_1 \stackrel{\text{def}}{=} \frac{\alpha_{32}}{\alpha_{31}}\pi_{22} + \frac{\alpha_{23}}{\alpha_{21}}\pi_{33} = \frac{\alpha_{23}}{\alpha_{21}}\left(\pi_{33} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right), \tag{3.20}$$

by (1.22), etc.,

$$f_1 = \pi_{11}(\pi_{23} + \pi_{32}) - \pi_{12}(\pi_{31} + \pi_{13}) - \pi_{13}\pi_{21}, \quad \text{etc.} \tag{3.21}$$

4. The Gradients of x_a, y_a

The twenty-seven relations given by the sets (1.24), (1.26), (1.29), (1.30), (1.32), (1.33), (2.2), (3.8) and (3.9) give the three components of the gradients of the nine functions π_{ab} in terms of the variables π_{ab}, x_a, y_a and u_a. The six conditions (3.15) and (3.16) give the six gradients $\frac{\delta u_1}{\delta e_1}, \frac{\delta u_2}{\delta e_2}, \frac{\delta u_3}{\delta e_3}, \frac{\delta u_1}{\delta e_3}, \frac{\delta u_2}{\delta e_1}, \frac{\delta u_3}{\delta e_2}$ in terms of these variables.

In this chapter we develop eighteen relations giving the three components of the gradients of the six functions x_a, y_a in terms of the variables π_{ab}, x_a, y_a, u_a and the three gradients, $\frac{\delta u_1}{\delta e_2}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$. It is apparent that no immediate expressions for these three gradients are available. We shall thus include them as additional variables, to be eliminated subsequently.

From (3.9) one has

$$\begin{aligned}\frac{\delta x_3}{\delta e_1} &= -\frac{\delta^2}{\delta e_1\,\delta e_2}\pi_{23} + \frac{\delta}{\delta e_1}(\pi_{13}\pi_{23} - \pi_{21}\pi_{33}) \\ &= -\frac{\delta^2}{\delta e_2\,\delta e_1}\pi_{23} + 2\pi_{13}\frac{\delta}{\delta e_3}\pi_{23} + \pi_{23}\left(\frac{\delta}{\delta e_2}\pi_{23} + \frac{\delta}{\delta e_1}\pi_{13}\right) \\ &\quad + \pi_{33}\left(\frac{\delta}{\delta e_3}\pi_{23} - \frac{\delta}{\delta e_1}\pi_{21}\right) - \pi_{21}\frac{\delta}{\delta e_1}\pi_{33}, \quad \text{by (1.11)}.\end{aligned}$$

Making the appropriate substitutions for the gradient terms we obtain*

$$\begin{aligned}\frac{\delta x_3}{\delta e_1} = &-\frac{\delta^2}{\delta e_2\,\delta e_1}\pi_{23} + 2\pi_{13}\frac{\delta}{\delta e_1}\pi_{23} + \pi_{23}(y_3 - x_3) + \frac{\alpha_{13}}{\alpha_{12}}\pi_{33}u_1 - \frac{\alpha_{12}}{\alpha_{13}}\pi_{21}u_3 \\ &+ \pi_{33}[-\pi_{22}(\pi_{13} + \pi_{31}) + \pi_{23}(2\pi_{12} + \pi_{21}) - \pi_{21}\pi_{32}], \quad \text{etc.}\end{aligned} \tag{4.1}$$

From $(3.6)^2$ and (1.35), one has

$$\frac{\delta y_3}{\delta e_1} = \frac{\delta^2}{\delta e_1\,\delta e_2}(\pi_{23} - \pi_{32}) - \frac{\delta}{\delta e_1}(\pi_{13}\pi_{23} - \pi_{13}\pi_{32}).$$

---

* Note that terms $\frac{\delta}{\delta e_1}\pi_{23}$ and $\frac{\delta^2}{\delta e_2\,\delta e_1}\pi_{23}$ may be expressed in terms of u_a and π_{ab} through (1.24) to (1.27). For reasons of economy of space such substitutions will be delayed until later in the analysis.

Commuting the first term by (1.11), substituting for the gradients and reducing, we obtain

$$\frac{\delta y_3}{\delta e_1} = \frac{\delta^2}{\delta e_2\, \delta e_1}(\pi_{23} - \pi_{32}) - 2\pi_{13}\frac{\delta}{\delta e_1}(\pi_{23} - \pi_{32})$$
$$+ (\pi_{32} - 2\pi_{23})\, y_3 - \pi_{33}x_2, \quad \text{etc.} \tag{4.2}$$

A check on (4.1) and (4.2) is obtained as follows. If they are added we obtain an expression for $\frac{\delta}{\delta e_1}(x_3 + y_3)$, which may be verified by taking the appropriate gradient of (3.10).

From (3.6)[1] and (1.35) one has

$$\frac{\delta x_3}{\delta e_2} = \frac{\delta^2}{\delta e_2\, \delta e_1}(\pi_{31} - \pi_{13}) + \frac{\delta}{\delta e_2}(\pi_{23}\pi_{31} - \pi_{23}\pi_{13}).$$

Commuting the first term by (1.11), substituting for the gradients and reducing, we obtain

$$\frac{\delta x_3}{\delta e_2} = \frac{\delta^2}{\delta e_1\, \delta e_2}(\pi_{31} - \pi_{13}) + 2\pi_{23}\frac{\delta}{\delta e_2}(\pi_{31} - \pi_{13})$$
$$- (\pi_{31} - 2\pi_{13})\, x_3 + \pi_{33}y_1, \quad \text{etc.} \tag{4.3}$$

From (3.8) one has

$$\frac{\delta y_3}{\delta e_2} = \frac{\delta^2}{\delta e_2\, \delta e_1}\pi_{13} + \frac{\delta}{\delta e_2}(\pi_{13}\pi_{23} - \pi_{12}\pi_{33}),$$

which similarly leads to

$$\frac{\delta y_3}{\delta e_2} = \frac{\delta^2}{\delta e_1\, \delta e_2}\pi_{13} + 2\pi_{23}\frac{\delta}{\delta e_2}\pi_{13} + \pi_{13}(y_3 - x_3) - \pi_{12}u_1 - \frac{\alpha_{32}}{\alpha_{31}}\pi_{33}u_3$$
$$+ \pi_{33}[\pi_{11}(\pi_{23} + \pi_{32}) - \pi_{13}(2\pi_{21} + \pi_{12}) + \pi_{12}\pi_{31}], \quad \text{etc.} \tag{4.4}$$

A check on (4.3) and (4.4) is obtained by taking the appropriate gradient of (3.10).

Again from (3.6) we have

$$\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2} = \left[\frac{\delta^2}{\delta e_3\, \delta e_2} - \frac{\delta^2}{\delta e_2\, \delta e_3}\right] \operatorname{div} e_1 - \frac{\delta}{\delta e_3}[\pi_{13} \operatorname{div} e_1] - \frac{\delta}{\delta e_2}[\pi_{12} \operatorname{div} e_1].$$

Applying the commutation formula (1.11), expanding, using (3.6) to eliminate the gradients of $\operatorname{div} e_1$, and using (3.10), we obtain the set of relations

$$\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2} = \pi_{11}\left[\frac{\delta}{\delta e_1}\operatorname{div} e_1 - (\operatorname{div} e_1)^2\right] + \operatorname{div} e_2 x_2 - \operatorname{div} e_3 y_3, \quad \text{etc.} \tag{4.5}$$

We note that by eliminating $\frac{\delta}{\delta e_2}\pi_{12}$ between (2.1) and (3.10) we obtain the formula

$$\frac{\delta}{\delta e_3}\pi_{13} = -(y_1 + x_1) + \pi_{21}\pi_{13} - \pi_{11}\pi_{32}, \quad \text{etc.} \tag{4.6}$$

This result also follows directly from (1.33) and (3.11). From (3.8) and (4.6) we have

$$\frac{\delta y_3}{\delta e_3}+\frac{\delta}{\delta e_1}(y_1+x_1)=\frac{\delta}{\delta e_3}\left[\frac{\delta}{\delta e_1}\pi_{13}+\pi_{13}\pi_{23}-\pi_{12}\pi_{33}\right]$$
$$+\frac{\delta}{\delta e_1}\left[-\frac{\delta}{\delta e_3}\pi_{13}+\pi_{21}\pi_{13}-\pi_{11}\pi_{32}\right].$$

Applying (1.11), expanding and eliminating $\frac{\delta}{\delta e_1}\pi_{13}$, $\frac{\delta}{\delta e_3}\pi_{13}$, $\frac{\delta}{\delta e_3}\pi_{23}$ and $\frac{\delta}{\delta e_1}\pi_{21}$ by (3.8), (4.6), (3.10) and (4.6) again, respectively, we obtain

$$\frac{\delta y_3}{\delta e_3}+\frac{\delta}{\delta e_1}(y_1+x_1)=-\operatorname{div}\boldsymbol{e}_1(y_1+x_1)-\operatorname{div}\boldsymbol{e}_3y_3$$
$$-\pi_{11}\left[\frac{\delta}{\delta e_1}\pi_{32}-\pi_{32}\operatorname{div}\boldsymbol{e}_1\right]-\pi_{22}\left[\frac{\delta}{\delta e_2}\pi_{13}-\pi_{13}\operatorname{div}\boldsymbol{e}_2\right]$$
$$-\pi_{33}\left[\frac{\delta}{\delta e_3}\pi_{12}-\pi_{12}\operatorname{div}\boldsymbol{e}_3\right],\quad\text{etc.}\tag{4.7}$$

Taking the directional derivative of (3.11) with respect to $\boldsymbol{e}_1$ and eliminating $\frac{\delta\pi_{11}}{\delta e_1},\frac{\delta\pi_{12}}{\delta e_1},\frac{\delta\pi_{13}}{\delta e_1},\frac{\delta\pi_{31}}{\delta e_1}$ and $\frac{\delta\pi_{21}}{\delta e_1}$ by (2.2), (3.9), (3.8), (1.32) and (1.33), respectively, we obtain

$$\frac{\delta}{\delta e_1}(y_1+x_1)=\frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1}-\pi_{11}\left[\frac{\delta}{\delta e_1}(\pi_{23}+\pi_{32})-2(\pi_{23}^2-\pi_{32}^2)\right]$$
$$-(\pi_{31}+\pi_{13})\,x_2+(\pi_{12}+\pi_{21})\,y_3-\frac{\alpha_{13}}{2\alpha_{12}}\pi_{13}u_1+\frac{\alpha_{21}}{2\alpha_{23}}\pi_{12}u_2$$
$$-\pi_{22}\pi_{13}^2+\pi_{33}\pi_{12}^2+2\pi_{12}\pi_{32}(\pi_{13}+\pi_{31})-2\pi_{13}\pi_{23}(\pi_{12}+\pi_{21}),\quad\text{etc.}\tag{4.8}$$

Eliminating $\frac{\delta}{\delta e_1}(y_1+x_1)$ between (4.7) and (4.8) and replacing y_1+x_1 in (4.7) by the expression (3.11) and then using (1.35), we obtain

$$\frac{\delta y_3}{\delta e_3}=-\frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1}+\pi_{11}\left[\frac{\delta}{\delta e_1}\pi_{23}-\pi_{23}(\pi_{23}-\pi_{32})\right]$$
$$+\pi_{22}\left[-\frac{\delta}{\delta e_2}\pi_{13}+\pi_{13}\pi_{31}\right]-\pi_{33}\left[\frac{\delta}{\delta e_3}\pi_{12}+\pi_{12}\pi_{21}\right]$$
$$+(\pi_{31}+\pi_{13})\,x_2-2\pi_{12}y_3+\frac{\alpha_{13}}{2\alpha_{12}}\pi_{13}u_1-\frac{\alpha_{21}}{2\alpha_{23}}\pi_{12}u_2$$
$$-\frac{\alpha_{32}}{2\alpha_{31}}(\pi_{23}-\pi_{32})\,u_3-\pi_{12}[(\pi_{13}+\pi_{31})(\pi_{23}+\pi_{32})-2\pi_{13}\pi_{23}]$$
$$+\pi_{13}\pi_{21}(\pi_{23}+\pi_{32}),\quad\text{etc.}\tag{4.9}$$

From (4.5) and (4.9) we now have, using (1.35),

$$\begin{aligned}\frac{\delta x_2}{\delta e_2} = & -\frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1} + \pi_{11}\left[\frac{\delta}{\delta e_1}\pi_{32} - \pi_{32}(\pi_{23} - \pi_{32})\right] \\ & + \pi_{22}\left[-\frac{\delta}{\delta e_2}\pi_{13} + \pi_{13}\pi_{31}\right] - \pi_{33}\left[\frac{\delta}{\delta e_3}\pi_{12} + \pi_{12}\pi_{21}\right] \\ & + 2\pi_{13}x_2 - (\pi_{12} + \pi_{12})\, y_3 + \frac{\alpha_{13}}{2\alpha_{12}}\pi_{13}u_1 \qquad (4.10) \\ & - \frac{\alpha_{21}}{2\alpha_{23}}\pi_{12}u_2 - \frac{\alpha_{32}}{2\alpha_{31}}(\pi_{23} - \pi_{32})\, u_3 \\ & - \pi_{12}[(\pi_{13} + \pi_{31})(\pi_{23} + \pi_{32}) - 2\pi_{13}\pi_{23}] + \pi_{13}\pi_{21}(\pi_{23} + \pi_{32}), \quad \text{etc.}\end{aligned}$$

The six sets of conditions, taken in pairs (4.1) and (4.4), (4.2) and (4.3), (4.9) and (4.10), give the eighteen gradient components of x_a and y_a. We have obtained these conditions independently by direct computation. The rotation condition serves as a check. By applying the transformation given by (1.13), (1.31) and (3.7) to (4.1), (4.2) and (4.9) we obtain (4.4), (4.3) and (4.10) as their respective duals.

5. Derivation of Integrals (1)

It appears that while the relatively simple expressions (3.15) and (3.16) are available for the gradients $\frac{\delta u_1}{\delta e_1}$, etc., and $\frac{\delta u_1}{\delta e_3}$, etc., the third set of gradients $\frac{\delta u_1}{\delta e_2}$, etc. is much more deeply entrenched. If one looks at (1.29) or (1.30), one sees that the first two sets appear as second order mixed gradients of the abnormalities, while the gradients $\frac{\delta u_1}{\delta e_2}$ depend on terms such as $\frac{\delta^2 \pi_{11}}{\delta e_2^2}$. Our purpose now is to generate expressions for the gradients $\frac{\delta u_1}{\delta e_2}$, *etc.* For purposes of reference we use the term "integral" to mean a polynomial relation among the eighteen variables π_{ab}, x_a, y_a, u_a and the three gradients $\frac{\delta u_1}{\delta e_2}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$.

Substituting directly the expressions from the sets (4.8), (4.9) and (4.10) into the identity

$$\frac{\delta}{\delta e_1}(y_1 + x_1) - \frac{\delta y_1}{\delta e_1} - \frac{\delta x_1}{\delta e_1} = 0,$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{\alpha_{21}}{\alpha_{23}}\frac{\delta u_2}{\delta e_3} + \frac{1}{2}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta u_3}{\delta e_1} + \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}\frac{\delta u_1}{\delta e_2} - 2\pi_{32}x_1 + 2\pi_{23}y_1 \\ + (\pi_{31} + \pi_{13})(y_2 - x_2) + (\pi_{12} + \pi_{21})(y_3 - x_3)\end{aligned}$$

$$+\frac{\alpha_{13}}{\alpha_{12}}(\pi_{31}-\pi_{13})\,u_1+\frac{\alpha_{21}}{\alpha_{23}}(\pi_{12}-\pi_{21})\,u_2+\frac{1}{2}\,\frac{\alpha_{32}}{\alpha_{31}}(\pi_{23}-\pi_{32})\,u_3$$
$$+\,2\pi_{11}(\pi_{23}^2-\pi_{32}^2)+\pi_{22}(\pi_{31}^2-\pi_{13}^2)+\pi_{33}(\pi_{12}^2-\pi_{21}^2)$$
$$+\,\pi_{31}\pi_{12}(2\pi_{23}+\pi_{32})-\pi_{13}[\pi_{32}(2\pi_{21}-\pi_{12})+\pi_{23}(\pi_{12}+\pi_{21})]=0. \tag{5.1}$$

In (5.1) we write

$$(\pi_{31}+\pi_{13})(y_2-x_2)=-2\pi_{13}x_2+2\pi_{31}y_2+(\pi_{13}-\pi_{31})(y_2+x_2),$$
$$(\pi_{12}+\pi_{21})(y_3-x_3)=-2\pi_{21}x_3+2\pi_{12}y_3+(\pi_{21}-\pi_{12})(y_3+x_3)$$

and substitute for y_2+x_2 and y_3+x_3 from the system (3.11). We obtain the symmetric relation

$$\frac{\alpha_{32}}{\alpha_{31}}\left[\frac{\delta u_3}{\delta e_1}+(\pi_{23}-\pi_{32})\,u_3\right]+\frac{\alpha_{13}}{\alpha_{12}}\left[\frac{\delta u_1}{\delta e_2}+(\pi_{31}-\pi_{13})\,u_1\right]$$
$$+\frac{\alpha_{21}}{\alpha_{23}}\left[\frac{\delta u_2}{\delta e_3}+(\pi_{12}-\pi_{21})\,u_2\right]$$
$$-\,4[(\pi_{32}x_1-\pi_{23}y_1)+(\pi_{13}x_2-\pi_{31}y_2)+(\pi_{21}x_3-\pi_{12}y_3)]$$
$$-\,4[\pi_{11}(\pi_{32}^2-\pi_{23}^2)+\pi_{22}(\pi_{13}^2-\pi_{31}^2)+\pi_{33}(\pi_{21}^2-\pi_{12}^2)]=0. \tag{5.2}$$

This symmetrical relation is invariant under the rotation given by (1.13), (1.31) and (3.7).

A second integral can be obtained as follows. From (3.8), (1.29), (1.30), (2.2), (3.15) and (3.16) we take the directional derivative of (3.15) with respect to e_3 to obtain

$$\frac{\delta^2 u_1}{\delta e_3\,\delta e_1}=-\frac{4\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_3}-4\frac{\alpha_{21}}{\alpha_{23}}u_2y_3+8\frac{\alpha_{21}}{\alpha_{23}}\pi_{12}\pi_{11}y_3-\frac{3}{2}\,\frac{\alpha_{13}}{\alpha_{12}}u_1^2+2y_2u_1$$
$$-\,[-\pi_{22}(3\pi_{13}+2\pi_{31})+3\pi_{21}(\pi_{23}+\pi_{32})+2\pi_{32}\pi_{12}]\,u_1$$
$$-\,(3\pi_{23}-2\pi_{32})\left[-4\pi_{33}x_1+(3\pi_{21}-2\pi_{12})\,u_1+\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}u_3\right]$$
$$+\frac{\alpha_{21}}{\alpha_{23}}\left[2\pi_{33}(\pi_{12}-\pi_{21})\,u_2-\pi_{33}\frac{\delta u_2}{\delta e_3}\right]. \tag{5.3}$$

Similarly, taking the directional derivative of (3.16) with respect to e_1, we obtain

$$\frac{\delta^2 u_1}{\delta e_1\,\delta e_3}=-4\pi_{33}\frac{\delta x_1}{\delta e_1}-4\frac{\alpha_{12}}{\alpha_{13}}u_3x_1-8\pi_{32}\pi_{33}x_1-\frac{3}{2}\,\frac{\alpha_{13}}{\alpha_{12}}u_1^2+2x_2u_1$$
$$+\,[\pi_{22}(3\pi_{31}+2\pi_{13})-3\pi_{23}(\pi_{12}+\pi_{21})-2\pi_{32}\pi_{12}]\,u_1$$
$$+\,(3\pi_{21}-2\pi_{12})\frac{\alpha_{21}}{\alpha_{23}}\left[-4\pi_{11}y_3-(3\pi_{23}-2\pi_{32})\frac{\alpha_{23}}{\alpha_{21}}u_1-\pi_{33}u_2\right]$$
$$+\frac{\alpha_{12}}{\alpha_{13}}\left[-2\pi_{11}(\pi_{23}-\pi_{32})\,u_3+\pi_{11}\frac{\delta u_3}{\delta e_1}\right]. \tag{5.4}$$

From the commutation formula (1.11) with (3.15) and (3.16), we have

$$\frac{\delta^2 u_1}{\delta e_1\, \delta e_3} - \frac{\delta^2 u_1}{\delta e_3\, \delta e_1} = \frac{\alpha_{21}}{\alpha_{23}} \pi_{12} \left[-4\pi_{11} y_3 - (3\pi_{23} - 2\pi_{32}) \frac{\alpha_{23}}{\alpha_{21}} u_1 - \pi_{33} u_2\right]$$
$$+ \pi_{22} \frac{\delta u_1}{\delta e_2} + \pi_{32} \left[-4\pi_{33} x_1 + (3\pi_{21} - 2\pi_{12})\, u_1 + \frac{\alpha_{12}}{\alpha_{13}} \pi_{11} u_3\right]. \tag{5.5}$$

From (5.3), (5.4) and (5.5) we now obtain

$$-4\pi_{33} \frac{\delta x_1}{\delta e_1} + 4 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11} \frac{\delta y_3}{\delta e_3} + \frac{\alpha_{12}}{\alpha_{13}} \pi_{11} \frac{\delta u_3}{\delta e_1} - \pi_{22} \frac{\delta u_1}{\delta e_2} + \frac{\alpha_{21}}{\alpha_{23}} \pi_{33} \frac{\delta u_2}{\delta e_3}$$
$$+ 2(x_2 - y_2)\, u_1 + 4 \frac{\alpha_{21}}{\alpha_{23}} u_2 y_3 - 4 \frac{\alpha_{12}}{\alpha_{13}} u_3 x_1 + \pi_{22}(\pi_{31} - \pi_{13})\, u_1$$
$$+ \frac{\alpha_{21}}{\alpha_{23}} \pi_{33}(\pi_{12} - \pi_{21})\, u_2 + \frac{\alpha_{12}}{\alpha_{13}} (\pi_{23} - \pi_{32})\, \pi_{11} u_3 - 4(3\pi_{23} - \pi_{32})\, \pi_{33} x_1$$
$$- 4 \frac{\alpha_{21}}{\alpha_{23}} (3\pi_{21} - \pi_{12})\, \pi_{11} y_3 = 0, \quad \text{etc.} \tag{5.6}$$

On substitution for $\dfrac{\delta x_1}{\delta e_1}$ and $\dfrac{\delta y_3}{\delta e_3}$ from the sets (4.9) and (4.10), and substituting for $\dfrac{\delta}{\delta e_1}\pi_{23}$, $\dfrac{\delta}{\delta e_1}\pi_{32}$, $\dfrac{\delta}{\delta e_2}\pi_{13}$ and $\dfrac{\delta}{\delta e_3}\pi_{12}$ from (1.24), (1.25), (1.26) and (1.27), we obtain the integral

$$3\pi_{33} \frac{\alpha_{21}}{\alpha_{23}} \frac{\delta u_2}{\delta e_3} + 3\pi_{11} \frac{\alpha_{12}}{\alpha_{13}} \frac{\delta u_3}{\delta e_1} - \pi_{22} \frac{\delta u_1}{\delta e_2} + 2(x_2 - y_2)\, u_1 - 4 \frac{\alpha_{12}}{\alpha_{13}} u_3 x_1$$
$$+ 4 \frac{\alpha_{21}}{\alpha_{23}} u_2 y_3 - 4\pi_{33}(\pi_{32} + 3\pi_{23})\, x_1 + 4 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}(\pi_{31} + \pi_{13})\, x_2$$
$$+ 4\pi_{33}(\pi_{31} + \pi_{13})\, y_2 - 4 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}(\pi_{12} + 3\pi_{21})\, y_3$$
$$+ \left[2 \left(\frac{\alpha_{13}}{\alpha_{12}} \pi_{33}\pi_{31} - \frac{\alpha_{31}}{\alpha_{32}} \pi_{11}\pi_{13}\right) + \pi_{22}(\pi_{31} - \pi_{13})\right] u_1$$
$$+ \left[3 \frac{\alpha_{21}}{\alpha_{23}} \pi_{33}(\pi_{12} - \pi_{21}) - 2 \left(\frac{\alpha_{21}}{\alpha_{23}}\right)^2 \pi_{11}\pi_{12}\right] u_2$$
$$+ \left[3 \frac{\alpha_{12}}{\alpha_{13}} \pi_{11}(\pi_{23} - \pi_{32}) - 2 \frac{\alpha_{32}}{\alpha_{31}} \pi_{33}\pi_{32}\right] u_3$$
$$+ 4\pi_{33}[\pi_{31}[(\pi_{32} + \pi_{23})\, (\pi_{12} + \pi_{21}) - 2\pi_{32}\pi_{12}] - \pi_{32}\pi_{13}(\pi_{12} + \pi_{21})]$$
$$- 4 \frac{\alpha_{21}}{\alpha_{23}} \pi_{11}[\pi_{12}[(\pi_{13} + \pi_{31})\, (\pi_{23} + \pi_{32}) - 2\pi_{13}\pi_{23}] - \pi_{13}\pi_{21}(\pi_{23} + \pi_{32})]$$

$$
\begin{aligned}
&- 8\pi_{33}^2\pi_{21}^2 + 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}\pi_{12}^2 + 4\left[\pi_{33}^2 - \frac{\alpha_{31}}{\alpha_{13}}\pi_{11}^2 + \frac{\alpha_{31}}{\alpha_{23}}\pi_{11}\pi_{22} - \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}\right]\pi_{12}\pi_{21} \\
&+ 4\pi_{33}\pi_{11}\pi_{32}^2 - 8\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2\pi_{23}^2 + 4\left[\frac{\alpha_{12}}{\alpha_{32}}\pi_{33}^2 - \pi_{33}\pi_{11} + \frac{\alpha_{13}}{\alpha_{23}}\pi_{33}\pi_{22} + \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2\right]\pi_{23}\pi_{32} \\
&- 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{22}\pi_{13}^2 - 4\pi_{22}\pi_{33}\pi_{31}^2 \\
&+ 4\left[-\frac{\alpha_{23}}{\alpha_{13}}\pi_{33}\pi_{11} + \pi_{22}\pi_{33} + \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{22} - \frac{\alpha_{21}\alpha_{21}}{\alpha_{23}\alpha_{31}}\pi_{11}\pi_{33}\right]\pi_{13}\pi_{31} \\
&+ \pi_{33}\left\{\frac{\alpha_{12}}{\alpha_{23}}\pi_{22}^3 + \frac{\alpha_{13}}{\alpha_{32}}\pi_{33}^3 + 4\pi_{11}\pi_{22}\pi_{33} + \left[\frac{\alpha_{23}}{\alpha_{21}} - 2\frac{\alpha_{12}}{\alpha_{32}} - \frac{\alpha_{13}\alpha_{31}}{\alpha_{21}\alpha_{32}}\right]\pi_{22}\pi_{33}^2\right. \\
&\left.+ \left[\frac{\alpha_{32}}{\alpha_{31}} - 2\frac{\alpha_{13}}{\alpha_{23}} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{31}\alpha_{23}}\right]\pi_{33}\pi_{22}^2 - 3\left(\frac{\alpha_{21}}{\alpha_{23}}\pi_{22} + \frac{\alpha_{31}}{\alpha_{32}}\pi_{33}\right)\pi_{11}^2\right\} \\
&+ \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left\{\frac{\alpha_{31}}{\alpha_{12}}\pi_{11}^3 + \frac{\alpha_{32}}{\alpha_{21}}\pi_{22}^3 + 4\pi_{11}\pi_{22}\pi_{33} + \left[\frac{\alpha_{21}}{\alpha_{23}} - 2\frac{\alpha_{32}}{\alpha_{12}} - \frac{\alpha_{13}\alpha_{31}}{\alpha_{12}\alpha_{23}}\right]\pi_{22}\pi_{11}^2\right. \\
&\left.+ \left[\frac{\alpha_{12}}{\alpha_{13}} - 2\frac{\alpha_{31}}{\alpha_{21}} - \frac{\alpha_{32}\alpha_{23}}{\alpha_{13}\alpha_{21}}\right]\pi_{11}\pi_{22}^2 - 3\left(\frac{\alpha_{13}}{\alpha_{12}}\pi_{11} + \frac{\alpha_{23}}{\alpha_{21}}\pi_{22}\right)\pi_{33}^2\right\} \\
&+ \pi_{11}\pi_{33}\left\{\frac{\alpha_{21}}{\alpha_{13}}\left(\frac{\alpha_{12}\alpha_{21}}{\alpha_{23}\alpha_{32}} - \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{32}} - 1\right)\pi_{11}^2 + \left(\frac{\alpha_{23}\alpha_{32}}{\alpha_{12}\alpha_{31}} - \frac{\alpha_{13}}{\alpha_{12}} - \frac{\alpha_{21}}{\alpha_{31}}\right)\pi_{33}^2\right. \\
&\left.- 2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{22} - 2\pi_{22}\pi_{33} + 2\frac{\alpha_{23}}{\alpha_{13}}\left(1 - \frac{\alpha_{12}\alpha_{21}}{\alpha_{32}\alpha_{23}}\right)\pi_{33}\pi_{11}\right\} = 0, \quad \text{etc.}
\end{aligned} \tag{5.7}
$$

If one applies the rotation transformation given by (1.13), (1.31) and (3.7) to the condition (5.7) one regains the same condition.

A second check on (5.6) and the relations leading to it is as follows. If one writes (5.6) as

$$
\begin{aligned}
-4\pi_{33}\frac{\delta x_1}{\delta e_1} + 4\pi_{11}\frac{\alpha_{21}}{\alpha_{23}}\frac{\delta y_3}{\delta e_3} + \frac{\alpha_{12}}{\alpha_{13}}\pi_{11}\frac{\delta u_3}{\delta e_1}& \\
- \pi_{22}\frac{\delta u_1}{\delta e_2} + \frac{\alpha_{21}}{\alpha_{23}}\pi_{33}\frac{\delta u_2}{\delta e_3} + \phi_{13} = 0&
\end{aligned} \tag{5.8}
$$

and eliminates the expression

$$
\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}\frac{\delta u_3}{\delta e_1} - \pi_{22}\frac{\delta u_1}{\delta e_2} + \frac{\alpha_{21}}{\alpha_{23}}\pi_{33}\frac{\delta u_2}{\delta e_3}
$$

between (5.8) and the condition obtained by permuting the indices forward once, one obtains by (1.22)

$$
4\pi_{11}\left(\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2}\right) + 4\left(\pi_{22}\frac{\alpha_{32}}{\alpha_{31}}\frac{\delta y_1}{\delta e_1} - \pi_{33}\frac{\alpha_{23}}{\alpha_{21}}\frac{\delta x_1}{\delta e_1}\right) + \phi_{21} + \frac{\alpha_{23}}{\alpha_{21}}\phi_{13} = 0. \tag{5.9}
$$

Since the conditions (5.8) are cyclic, the conditions (5.9) can represent at most only two independent conditions. If one eliminates $\frac{\delta y_3}{\delta e_3} - \frac{\delta x_2}{\delta e_2}$ from (5.9) by means of (4.5) one obtains the same condition as that obtained by taking the directional derivative with respect to e_1 of the presented member of the set (3.18). This latter condition is only equivalent to two conditions because the three conditions (3.18) are the components of the curl of the vector given by (2.6) so that the divergence must vanish.

6. Derivation of Integrals (2)

Consider the member of the set (3.18)*

$$\pi_{13}\frac{\alpha_{12}}{\alpha_{13}}u_3 + \pi_{23}u_1 + 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}y_3 - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}x_3\right] + 2\pi_{33}^2 \operatorname{div} e_3 = 0. \tag{6.1}$$

We take the directional derivative of (6.1) with respect to e_1 using (1.35), (3.8), (2.2), (1.29), (1.30) and (3.6). We obtain

$$\begin{aligned}
&(y_3 - \pi_{13}\pi_{23} + \pi_{12}\pi_{33})\frac{\alpha_{12}}{\alpha_{13}}u_3 + \pi_{13}\frac{\alpha_{12}}{\alpha_{13}}\frac{\delta u_3}{\delta e_1} + \left(\frac{\delta}{\delta e_1}\pi_{23}\right)u_1 + \pi_{23}\frac{\delta u_1}{\delta e_1} \\
&\quad + 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{32}(2\pi_{11}y_3) + 2\pi_{23}\left(-\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}y_3 + \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}x_3\right) - \frac{\alpha_{12}}{\alpha_{13}}u_3x_3\right] \\
&\quad + 4\pi_{33}\left(\frac{\alpha_{12}}{\alpha_{13}}u_3 + 2\pi_{32}\pi_{33}\right)\operatorname{div} e_3 + 2\pi_{33}^2(y_2 + \pi_{32}\operatorname{div} e_3) \\
&\quad + 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_1} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\frac{\delta x_3}{\delta e_1}\right] = 0.
\end{aligned} \tag{6.2}$$

Eliminating the term $\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}y_3 - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}x_3$ from (6.2) by means of (6.1) we obtain

$$\begin{aligned}
&[y_3 - 2x_3 + \pi_{13}\pi_{23} + \pi_{33}(5\pi_{12} - 4\pi_{21})]\frac{\alpha_{12}}{\alpha_{13}}u_3 + \left[\frac{\delta}{\delta e_1}\pi_{23} + 2\pi_{23}^2\right]u_1 \\
&\quad + 2\pi_{33}^2y_2 + 4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{32}y_3 + 2\pi_{33}^2 \operatorname{div} e_3\,[2\pi_{23} + 5\pi_{32}] \\
&\quad + 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_1} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\frac{\delta x_3}{\delta e_1}\right] + \pi_{13}\frac{\alpha_{12}}{\alpha_{13}}\frac{\delta u_3}{\delta e_1} + \pi_{23}\frac{\delta u_1}{\delta e_1} = 0, \quad \text{etc.}
\end{aligned} \tag{6.3}$$

* Note that we avoid the operation discussed at the end of Chapter 5.

Taking the directional derivative of (6.1) with respect to e_2 in a similar manner we get

$$[-x_3 + 2y_3 - \pi_{13}\pi_{23} + \pi_{33}(4\pi_{12} - 5\pi_{21})]\, u_1$$
$$+ \left[\frac{\delta}{\delta e_2}\pi_{13} - 2\pi_{13}^2\right]\frac{\alpha_{12}}{\alpha_{13}} u_3 + 2\pi_{33}^2 x_1 + 4\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{31}x_3 - 2\pi_{33}^2 \operatorname{div} e_3\, [2\pi_{13} + 5\pi_{31}]$$
$$+ 2\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\frac{\delta y_3}{\delta e_2} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\frac{\delta x_3}{\delta e_2}\right] + \pi_{13}\frac{\alpha_{12}}{\alpha_{13}}\frac{\delta u_3}{\delta e_2} + \pi_{23}\frac{\delta u_1}{\delta e_2} = 0, \quad \text{etc.} \tag{6.4}$$

The conditions (6.3) and (6.4) have yet to be expressed explicitly in terms of the variables π_{ab}, x_a, y_a, u_a and the three gradients $\frac{\delta u_1}{\delta e_2}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$.

The gradients $\frac{\delta u_1}{\delta e_1}$ and $\frac{\delta u_3}{\delta e_2}$ are readily eliminated from (6.3) and (6.4) by substitution from (3.15) and (3.16). Likewise, we eliminate the gradients $\frac{\delta}{\delta e_1}\pi_{23}$ and $\frac{\delta}{\delta e_2}\pi_{13}$ by the relations (1.24) to (1.27). To eliminate $\frac{\delta x_3}{\delta e_1}, \frac{\delta y_3}{\delta e_1}, \frac{\delta x_3}{\delta e_2}$ and $\frac{\delta y_3}{\delta e_2}$ we use (4.1), (4.2), (4.3) and (4.4). In making this substitution expressions for $\frac{\delta}{\delta e_1}\pi_{23}$, $\frac{\delta}{\delta e_1}\pi_{32}$, $\frac{\delta}{\delta e_2}\pi_{31}$, $\frac{\delta}{\delta e_2}\pi_{13}$ are obtained directly from (1.24) to (1.27). Expressions for the second order mixed derivatives in (4.1) to (4.6) are naturally more complicated, but they are all obtainable by taking the appropriate gradients of the relations (1.24) to (1.27). We illustrate by considering $\frac{\delta^2}{\delta e_2\,\delta e_1}\pi_{23}$ occurring in the expression (4.1) for $\frac{\delta x_3}{\delta e_1}$. We have

$$\frac{\delta^2}{\delta e_2\,\delta e_1}\pi_{23} = -2\pi_{23}\frac{\delta}{\delta e_2}\pi_{23} + \frac{\alpha_{32}}{\alpha_{12}}\left(\pi_{21}\frac{\delta}{\delta e_2}\pi_{12} + \pi_{12}\frac{\delta}{\delta e_2}\pi_{21}\right)$$
$$+ \frac{\partial\eta_{123}}{\partial\pi_{11}}\frac{\delta\pi_{11}}{\delta e_2} + \frac{\partial\eta_{123}}{\partial\pi_{22}}\frac{\delta\pi_{22}}{\delta e_2} + \frac{\partial\eta_{123}}{\partial\pi_{33}}\frac{\delta\pi_{33}}{\delta e_2}, \tag{6.5}$$

where $\frac{\delta}{\delta e_2}\pi_{23}, \frac{\delta}{\delta e_2}\pi_{12}, \frac{\delta}{\delta e_2}\pi_{21}, \frac{\delta\pi_{11}}{\delta e_2}, \frac{\delta\pi_{22}}{\delta e_2}, \frac{\delta\pi_{33}}{\delta e_2}$ are given respectively by (3.9), (1.32), (3.8), (1.29), (2.2) and (1.30), while by (1.27)

$$\frac{\partial\eta_{123}}{\partial\pi_{11}} = \frac{1}{2}\left(-\frac{\alpha_{32}}{\alpha_{12}}\pi_{22} + \frac{\alpha_{31}}{\alpha_{12}}\pi_{11}\right), \tag{6.6}$$

$$\frac{\partial\eta_{123}}{\partial\pi_{22}} = \frac{1}{2}\left(\pi_{33} - \frac{\alpha_{32}}{\alpha_{12}}\pi_{11} + \frac{\alpha_{32}}{\alpha_{12}}\left(\frac{\alpha_{32}}{\alpha_{12}} - \frac{\alpha_{21}}{\alpha_{23}}\right)\pi_{22}\right), \tag{6.7}$$

$$\frac{\partial\eta_{123}}{\partial\pi_{33}} = \frac{1}{2}\left(\pi_{22} - 3\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}\right). \tag{6.8}$$

After a somewhat tedious but straightforward calculation we obtain from (6.3) (6.4), respectively, the two sets of integrals

$$
\begin{aligned}
&\frac{\alpha_{12}}{\alpha_{13}}\pi_{13}\frac{\delta u_3}{\delta e_1}+\frac{\alpha_{12}}{\alpha_{13}}(y_3-2x_3)\,u_3+2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}(3\pi_{32}-2\pi_{23})\,y_3\\
&+2\left(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right)\left(2\pi_{23}x_3+\frac{\alpha_{32}}{\alpha_{12}}\pi_{12}y_1\right)+2\pi_{33}\left(\pi_{33}y_2-\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}x_2\right)\\
&+\left\{\frac{5}{4}\frac{\alpha_{31}}{\alpha_{12}}\pi_{11}^2+\frac{5}{4}\left(\frac{\alpha_{12}}{\alpha_{13}}-\frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{13}}\right)\pi_{22}^2-\frac{3}{4}\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}^2+\left[2\frac{\alpha_{21}}{\alpha_{23}}-\frac{5}{2}\frac{\alpha_{32}}{\alpha_{12}}\right]\pi_{11}\pi_{22}\right.\\
&\left.+2\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}\pi_{33}-\frac{9}{2}\pi_{22}\pi_{33}+2\pi_{23}\pi_{32}+\frac{\alpha_{32}}{\alpha_{12}}\pi_{12}\pi_{21}\right\}u_1\\
&+\frac{\alpha_{21}}{\alpha_{23}}\left[2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{32}-\left(\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}+\pi_{33}\right)\pi_{23}\right]u_2+\left\{\left(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right)\frac{\alpha_{32}\alpha_{32}}{\alpha_{12}\alpha_{31}}\pi_{21}\right.\\
&\left.+\frac{\alpha_{12}}{\alpha_{13}}(3\pi_{13}\pi_{23}+\pi_{33}(5\pi_{12}-4\pi_{21}))+2\left(\frac{\alpha_{12}}{\alpha_{13}}\right)^2\pi_{22}\pi_{21}\right\}u_3\\
&+2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left\{-\frac{\alpha_{32}}{\alpha_{12}}[(\pi_{12}-\pi_{21})\,\pi_{13}+\pi_{12}\pi_{31}]\,\pi_{21}-\frac{\alpha_{23}}{\alpha_{13}}\pi_{13}\pi_{31}(\pi_{13}+\pi_{31})\right.\\
&\left.+(\pi_{23}\pi_{32}+2\pi_{32}^2)\,\pi_{13}+\left[2\pi_{32}(\pi_{23}+\pi_{32})+\frac{\alpha_{32}}{\alpha_{12}}\pi_{12}\pi_{21}\right]\pi_{31}\right\}\\
&-2\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\left\{\frac{\alpha_{32}}{\alpha_{12}}[(\pi_{12}-\pi_{21})\,\pi_{13}+\pi_{12}\pi_{31}]\,\pi_{21}+\pi_{23}[(\pi_{23}+\pi_{32})\,\pi_{31}+\pi_{32}\pi_{13}]\right\}\\
&+\left[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\right]\left[4\pi_{33}\pi_{23}\pi_{21}+2\frac{\alpha_{32}}{\alpha_{12}}\pi_{11}(\pi_{12}\pi_{23}-\pi_{21}\pi_{32})\right]\\
&-4\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}\pi_{12}\pi_{32}+2\pi_{33}^2(4\pi_{23}+5\pi_{32})\,(\pi_{12}-\pi_{21})\\
&-2\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{33}(\pi_{23}\pi_{12}-\pi_{21}\pi_{32})\\
&+\left\{\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left[\left(\frac{\alpha_{31}}{\alpha_{12}}-\frac{1}{2}\frac{\alpha_{21}}{\alpha_{13}}\right)\pi_{11}^2-\left(\frac{3}{2}\frac{\alpha_{12}}{\alpha_{13}}+\frac{\alpha_{32}\alpha_{23}}{\alpha_{21}\alpha_{13}}\right)\pi_{22}^2\right.\right.\\
&\left.+\left(\frac{5}{2}\frac{\alpha_{13}}{\alpha_{12}}+\frac{1}{2}\frac{\alpha_{23}\alpha_{32}}{\alpha_{12}\alpha_{31}}\right)\pi_{33}^2-2\frac{\alpha_{32}}{\alpha_{12}}\pi_{11}\pi_{22}-\pi_{22}\pi_{33}+\frac{\alpha_{23}}{\alpha_{13}}\pi_{33}\pi_{11}\right]\\
&+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\left[\frac{\alpha_{31}}{\alpha_{12}}\pi_{11}^2+\left(\frac{\alpha_{12}}{\alpha_{13}}-\frac{\alpha_{32}\alpha_{23}}{\alpha_{21}\alpha_{13}}\right)\pi_{22}^2+3\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}^2\right.\\
&\left.\left.-2\frac{\alpha_{32}}{\alpha_{12}}\pi_{11}\pi_{22}+2\pi_{22}\pi_{33}\right]\right\}\pi_{13}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl\{\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\Biggl[\frac{1}{2}\Bigl(\frac{\alpha_{31}}{\alpha_{12}}-\frac{\alpha_{21}}{\alpha_{13}}\Bigr)\pi_{11}^2+\Bigl(4\,\frac{\alpha_{12}}{\alpha_{13}}+\frac{3}{2}\,\frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{13}}\Bigr)\pi_{22}^2\\
&+\Bigl(4\,\frac{\alpha_{13}}{\alpha_{12}}+\frac{1}{2}\,\frac{\alpha_{23}\alpha_{32}}{\alpha_{12}\alpha_{31}}\Bigr)\pi_{33}^2+\frac{\alpha_{32}}{\alpha_{12}}\pi_{11}\pi_{22}-8\pi_{22}\pi_{33}+\frac{\alpha_{23}}{\alpha_{13}}\pi_{33}\pi_{11}\Biggr]\\
&+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\Biggl[2\Bigl(-\frac{\alpha_{12}}{\alpha_{13}}+\frac{\alpha_{32}\alpha_{23}}{\alpha_{21}\alpha_{13}}\Bigr)\pi_{22}^2+6\,\frac{\alpha_{13}}{\alpha_{12}}\pi_{33}^2\\
&+2\,\frac{\alpha_{32}}{\alpha_{12}}\pi_{11}\pi_{22}-2\pi_{22}\pi_{33}\Biggr]\Biggr\}\pi_{31}=0,\quad \text{etc.}
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
&\pi_{23}\,\frac{\delta u_1}{\delta e_2}+(-x_3+2y_3)\,u_1+2\,\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}(3\pi_{31}-2\pi_{13})\,x_3\\
&+2\Bigl(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\Bigr)\Bigl(2\pi_{13}y_3+\frac{\alpha_{31}}{\alpha_{21}}\pi_{21}x_2\Bigr)\\
&+2\pi_{33}\Bigl(\pi_{33}x_1-\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}y_1\Bigr)+\frac{\alpha_{23}}{\alpha_{21}}\Biggl[-\Bigl[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\Bigr]\frac{\alpha_{31}\alpha_{31}}{\alpha_{32}\alpha_{21}}\pi_{12}\\
&-\frac{\alpha_{21}}{\alpha_{23}}\,[3\pi_{23}\pi_{13}+\pi_{33}(5\pi_{21}-4\pi_{12})]-2\Bigl(\frac{\alpha_{21}}{\alpha_{23}}\Bigr)^2\pi_{11}\pi_{12}\Biggr]\,u_1\\
&+\frac{\alpha_{21}}{\alpha_{23}}\Biggl[2\,\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{31}-\Bigl(\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\pi_{33}\Bigr)\pi_{13}\Biggr]\,u_2\\
&+\frac{\alpha_{12}}{\alpha_{13}}\Biggl[-\frac{5}{4}\,\frac{\alpha_{32}}{\alpha_{21}}\pi_{22}^2+\frac{3}{4}\,\frac{\alpha_{23}}{\alpha_{21}}\pi_{33}^2-\frac{5}{4}\Bigl(\frac{\alpha_{21}}{\alpha_{23}}-\frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}}\Bigr)\pi_{11}^2\\
&-2\,\frac{\alpha_{32}}{\alpha_{31}}\,\pi_{22}\pi_{33}+\frac{9}{2}\,\pi_{33}\pi_{11}-\Bigl(2\frac{\alpha_{12}}{\alpha_{13}}-\frac{5}{2}\,\frac{\alpha_{31}}{\alpha_{21}}\Bigr)\pi_{11}\pi_{22}-\frac{\alpha_{31}}{\alpha_{21}}\pi_{21}\pi_{12}-2\pi_{31}\pi_{13}\Biggr]\,u_3\\
&+2\,\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\Biggl[-\frac{\alpha_{31}}{\alpha_{21}}\,[(\pi_{21}-\pi_{12})\,\pi_{23}+\pi_{21}\pi_{32}]\,\pi_{12}\\
&-\frac{\alpha_{13}}{\alpha_{23}}\pi_{23}\pi_{32}(\pi_{23}+\pi_{32})+(\pi_{13}\pi_{31}+2\pi_{31}^2)\,\pi_{23}\\
&+\Bigl(2\pi_{31}(\pi_{31}+\pi_{13})+\frac{\alpha_{31}}{\alpha_{21}}\pi_{21}\pi_{12}\Bigr)\pi_{32}\Biggr]\\
&-2\,\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\Biggl[\frac{\alpha_{31}}{\alpha_{21}}\,[(\pi_{21}-\pi_{12})\,\pi_{23}+\pi_{21}\pi_{32}]\,\pi_{12}+\pi_{13}\,[(\pi_{31}+\pi_{13})\,\pi_{32}+\pi_{31}\pi_{23}]\Biggr]\\
&+\Bigl[\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}+\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\Bigr]\Bigl[4\pi_{33}\pi_{13}\pi_{12}+2\,\frac{\alpha_{31}}{\alpha_{21}}\pi_{22}(\pi_{21}\pi_{13}-\pi_{12}\pi_{31})\Bigr]\\
&-4\,\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\pi_{33}\pi_{31}\pi_{21}+2\pi_{33}^2(4\pi_{13}+5\pi_{31})\,(\pi_{21}-\pi_{12})
\end{aligned}
$$

$$
\begin{aligned}
&- 2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\pi_{33}(\pi_{13}\pi_{21} - \pi_{31}\pi_{12}) \\
&+ \left\{\frac{\alpha_{12}}{\alpha_{13}}\pi_2 \left[\left(\frac{\alpha_{32}}{\alpha_{21}} - \frac{1}{2}\frac{\alpha_{12}}{\alpha_{23}}\right)\pi_{22}^2 + \left(\frac{5}{2}\frac{\alpha_{23}}{\alpha_{21}} + \frac{1}{2}\frac{\alpha_{31}\alpha_{13}}{\alpha_{21}\alpha_{32}}\right)\pi \right.\right. \\
&- \left.\left(\frac{3}{2}\frac{\alpha_{21}}{\alpha_{23}} + \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}}\right)\pi_{11}^2 + \frac{\alpha_{13}}{\alpha_{23}}\pi_{22}\pi_{33} - \pi_{33}\pi_{11} - 2\frac{\alpha_{31}}{\alpha_{21}}\pi_{11}\pi_{22}\right] \\
&+ \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left[\frac{\alpha_{32}}{\alpha_{21}}\pi_{22}^2 + 3\frac{\alpha_{23}}{\alpha_{21}}\pi_{33}^2 + \left(\frac{\alpha_{21}}{\alpha_{23}} - \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}}\right)\pi_{11}^2 + 2\pi_{33}\pi_{11}\right. \\
&- \left.\left.2\frac{\alpha_{31}}{\alpha_{21}}\pi_{11}\pi_{22}\right]\right\}\pi_{23} \\
&+ \left\{\frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\left[\frac{1}{2}\left(\frac{\alpha_{32}}{\alpha_{21}} - \frac{\alpha_{12}}{\alpha_{23}}\right)\pi_{22}^2 + \left(4\frac{\alpha_{23}}{\alpha_{21}} + \frac{1}{2}\frac{\alpha_{31}\alpha_{13}}{\alpha_{21}\alpha_{32}}\right)\pi_{33}^2\right.\right. \\
&+ \left.\left(4\frac{\alpha_{21}}{\alpha_{23}} + \frac{3}{2}\frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}}\right)\pi_{11}^2 + \frac{\alpha_{13}}{\alpha_{23}}\pi_{22}\pi_{33} - 8\pi_{33}\pi_{11} + \frac{\alpha_{31}}{\alpha_{21}}\pi_{11}\pi_{22}\right] \\
&+ \frac{\alpha_{21}}{\alpha_{23}}\pi_{11}\left[6\frac{\alpha_{23}}{\alpha_{21}}\pi_{33}^2 + 2\left(-\frac{\alpha_{21}}{\alpha_{23}} + \frac{\alpha_{31}\alpha_{13}}{\alpha_{23}\alpha_{12}}\right)\pi_{11}^2 - 2\pi_{33}\pi_{11}\right. \\
&+ \left.\left.2\frac{\alpha_{31}}{\alpha_{21}}\pi_{11}\pi_{22}\right]\right\}\pi_{32} = 0, \quad \text{etc.}
\end{aligned}
\tag{6.10}
$$

While the conditions (6.9) and (6.10) were worked out independently, it may easily be verified that they are duals. One may obtain (6.10) by applying the rotation transformation given by (1.13), (1.31) and (3.7) to the condition (6.9).

7. Proof of Main Theorem 1

With the six quantities x_a and y_a given in terms of u_1, u_2 and u_3 by (3.19), we have ten algebraic relations among the variables $\frac{\delta u_1}{\delta e_2}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$, u_1, u_2, u_3, the six curvatures π_{ab}, the three abnormalities π_{aa} and the two independent ratios $\frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}$. These ten relations are the symmetrical condition (5.2), the three conditions (5.7) and the six conditions given by (6.9) and (6.10).

If these conditions are independent at each level of the elimination, we can in principle eliminate $\frac{\delta u_1}{\delta e_2}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$, u_1, u_2, u_3 to obtain four homogeneous polynomials

$$
F_\alpha\left(\pi_{aa}, \pi_{ab}, \frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_2^2}{\sigma_3^2}\right) = 0, \quad \alpha = 1 \ldots 4. \tag{7.1}
$$

We are then faced with the problem of determining whether the conditions (7.1) are independent. For example, they could possess a symmetrical common factor. It may be possible to resolve these immense difficulties using symbolic computation by computer (1979 [1]).

We can, however, prove that there exists at least *one* homogeneous polynomial of the type (7.1). We do this by substituting appropriate values for π_{aa}, π_{ab}, σ_1^2, σ_2^2, σ_3^2, and performing the elimination. Since there are redundancies, there is evidently more than one way of doing this elimination. We obtain a series of necessary consequences of the expressions with numbers representing complicated polynomial functions of the above variables and we finally obtain a non-vanishing numerical value for the desired functional relation of the form (7.1). There are of course many ways of achieving such a representation since we have the redundancy in the integrals at the beginning. We are, however, seeking just one representation. In choosing the values for the variables we must be careful that the order of the equations is not reduced at any stage of the elimination.

We choose $\pi_{11} = 1$, $\pi_{22} = 2$, $\pi_{33} = -1$, and all $\pi_{ab} = 1$, $a \neq b$. We take

$$\sigma_1^2 : \sigma_2^2 : \sigma_3^2 = 1 : 2 : 3,^\star$$

so that

$$\begin{aligned} &\alpha_{12} = \frac{1}{2}, \quad \alpha_{23} = \frac{1}{3}, \quad \alpha_{31} = -2, \\ &\alpha_{21} = -1, \quad \alpha_{32} = -\frac{1}{2}, \quad \alpha_{13} = \frac{2}{3}. \end{aligned} \tag{7.2}$$

From (3.19) we have

$$\begin{aligned} x_1 &= \frac{1}{5}\left[-u_1 + 3u_2 + \frac{3}{8}u_3 + 3\right], \\ y_1 &= -\frac{1}{5}\left[-u_1 + 3u_2 - \frac{u_3}{4} - 2\right], \\ x_2 &= \frac{1}{8}\left[-\frac{8}{3}u_1 + 6u_2 + \frac{3}{2}u_3 + 4\right], \\ y_2 &= -\frac{1}{8}\left[-8u_1 + 6u_2 + \frac{3}{2}u_3 + 12\right], \\ x_3 &= -\frac{u_1}{3} - 3u_2 - \frac{u_3}{4} + 10, \\ y_3 &= \frac{u_1}{3} + \frac{3}{2}u_2 + \frac{u_3}{4} - 5. \end{aligned} \tag{7.3}$$

$\star$ In accordance with (1.5) this means $\sigma_1^2 = \frac{1}{\sqrt[3]{6}}$, $\sigma_2^2 = \frac{2}{\sqrt[3]{6}}$, $\sigma_3^2 = \frac{3}{\sqrt[3]{6}}$.

but by (8.2)

$$u_2 = -2\frac{\alpha_{32}}{\alpha_{31}}\pi_{12}\pi_{22},$$

so that by (1.22), since π_{12} does not vanish,

$$\pi_{33} - \frac{1}{2}\frac{\alpha_{12}}{\alpha_{13}}\pi_{22} = 0. \tag{8.16}$$

Again, by $(8.2)^2$

$$\frac{\pi_{32}}{\pi_{23}} = -\frac{\alpha_{13}}{\alpha_{12}}\frac{\pi_{33}}{\pi_{22}}$$

so that by (8.14)

$$\pi_{33} + \frac{1}{2}\frac{\alpha_{12}}{\alpha_{13}}\pi_{22} = 0. \tag{8.17}$$

By (8.16) and (8.17) $\pi_{22} = \pi_{33} = 0$, which is a contradiction. This completes the proof of Lemma 8.1.

From Lemma 8.1 and (8.3) we immediately have

Lemma 8.2. *If one or more of the three curvatures* $\pi_{21}, \pi_{32}, \pi_{13}$ *vanishes, there are no new solutions.*

We now prove

Lemma 8.3. *If there are new solutions when the ratios of the abnormalities are constant, then the ratios* $\frac{\pi_{12}}{\pi_{11}}, \frac{\pi_{23}}{\pi_{11}}, \frac{\pi_{31}}{\pi_{11}}$ *and* $\frac{\pi_{21}}{\pi_{11}}, \frac{\pi_{32}}{\pi_{11}}, \frac{\pi_{13}}{\pi_{11}}$ *must all be constant.*

From (8.2) the ratio $\frac{\pi_{13}}{\pi_{31}}$ is constant, so that

$$\pi_{31}\frac{\delta}{\delta e_2}\pi_{13} - \pi_{31}\frac{\delta}{\delta e_2}\pi_{31} = 0.$$

Substituting for π_{31} from (8.2) and cancelling π_{13}, which does not vanish, we obtain

$$\frac{\pi_{33}}{\pi_{11}}\frac{\delta}{\delta e_2}\pi_{13} + \frac{\alpha_{21}}{\alpha_{23}}\frac{\delta}{\delta e_2}\pi_{31} = 0. \tag{8.18}$$

Substituting from (1.24), (1.25), (1.26) and (1.27) into (8.18), using $\pi_{13} = -\frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}}\pi_{31}$, *etc.*, as given by (8.2) and using (1.22), we obtain from (8.18)

$$\frac{\alpha_{21}}{\alpha_{23}}\left[\frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}} - 1\right]\theta_1^2 + \frac{\alpha_{32}}{\alpha_{21}}\frac{\pi_{22}\pi_{33}}{\pi_{11}^2}\theta_2^2 + \frac{\alpha_{31}\alpha_{13}}{\alpha_{32}\alpha_{23}}\frac{\pi_{33}}{\pi_{22}}\theta_3^2 + \frac{\pi_{33}}{\pi_{11}^3}\xi_{213} + \frac{\alpha_{21}}{\alpha_{23}}\frac{\eta_{231}}{\pi_{11}^2} = 0 \tag{8.19}$$

where

$$\theta_1 = \frac{\pi_{31}}{\pi_{11}}, \quad \theta_2 = \frac{\pi_{12}}{\pi_{11}}, \quad \theta_3 = \frac{\pi_{23}}{\pi_{11}}.$$

We have three equations like (8.19) in θ_1^2, θ_2^2, θ_3^2. Evaluating the discriminant of these equations using (1.22), one has, since the abnormalities are non-vanishing and can be cancelled,

$$D = \begin{vmatrix} \frac{\alpha_{21}}{\alpha_{23}}\left[\frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}} - 1\right] & \frac{\alpha_{32}}{\alpha_{21}}\frac{\pi_{22}\pi_{33}}{\pi_{11}^2} & \frac{\alpha_{31}\alpha_{13}}{\alpha_{32}\alpha_{23}}\frac{\pi_{33}}{\pi_{22}} \\ \frac{\alpha_{12}\alpha_{21}}{\alpha_{13}\alpha_{31}}\frac{\pi_{11}}{\pi_{33}} & \frac{\alpha_{32}}{\alpha_{31}}\left[\frac{\alpha_{32}}{\alpha_{31}}\frac{\pi_{22}}{\pi_{11}} - 1\right] & \frac{\alpha_{13}}{\alpha_{32}}\frac{\pi_{33}\pi_{11}}{\pi_{22}^2} \\ \frac{\alpha_{21}}{\alpha_{13}}\frac{\pi_{11}\pi_{22}}{\pi_{33}^2} & \frac{\alpha_{23}\alpha_{32}}{\alpha_{21}\alpha_{12}}\frac{\pi_{22}}{\pi_{11}} & \frac{\alpha_{13}}{\alpha_{12}}\left[\frac{\alpha_{13}}{\alpha_{12}}\frac{\pi_{33}}{\pi_{22}} - 1\right] \end{vmatrix}$$

$$= -\left[-2 + \frac{\alpha_{31}}{\alpha_{32}}\frac{\pi_{11}}{\pi_{22}} + \frac{\alpha_{12}}{\alpha_{13}}\frac{\pi_{22}}{\pi_{33}} + \frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}} + \frac{\alpha_{23}}{\alpha_{21}}\frac{\pi_{33}}{\pi_{11}} + \frac{\alpha_{13}}{\alpha_{12}}\frac{\pi_{33}}{\pi_{22}} + \frac{\alpha_{32}}{\alpha_{31}}\frac{\pi_{22}}{\pi_{11}}\right]$$

$$+ \frac{\alpha_{31}}{\alpha_{32}}\frac{\pi_{11}}{\pi_{22}} + \frac{\alpha_{13}}{\alpha_{12}}\frac{\pi_{33}}{\pi_{22}} + 1 + \frac{\alpha_{23}}{\alpha_{21}}\frac{\pi_{33}}{\pi_{11}} + \frac{\alpha_{32}}{\alpha_{31}}\frac{\pi_{22}}{\pi_{11}} + 1 + \frac{\alpha_{12}}{\alpha_{13}}\frac{\pi_{22}}{\pi_{33}} + \frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}} = 4. \tag{8.20}$$

Again the terms $\frac{\pi_{33}}{\pi_{11}^3}\xi_{213} + \frac{\alpha_{21}}{\alpha_{23}}\frac{\eta_{231}}{\pi_{11}^2}$ in (8.19) involve only the ratios of the abnormalities and are constant. We see that equations (8.19) may be solved to give constant values of θ_1^2, θ_2^2 and θ_3^2. Thus the ratios $\frac{\pi_{12}}{\pi_{11}}, \frac{\pi_{23}}{\pi_{11}}, \frac{\pi_{31}}{\pi_{11}}$ are constant.

Finally, from (8.2)

$$\frac{\pi_{13}}{\pi_{11}} = -\frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{31}}{\pi_{11}}\left(\frac{\pi_{11}}{\pi_{33}}\right), \quad \text{etc.}$$

so the ratios $\frac{\pi_{21}}{\pi_{11}}, \frac{\pi_{32}}{\pi_{11}}, \frac{\pi_{13}}{\pi_{11}}$ are constant. This proves Lemma 8.3.

We prove

Lemma 8.4. *If there are new solutions for the case under consideration, then at least one of the quantities λ_1, λ_2, λ_3, defined in (2.7) must be zero.*

Since $\frac{\pi_{21}}{\pi_{11}}$ is constant, $\frac{\delta}{\delta e_1}\frac{\pi_{21}}{\pi_{11}} = 0$, and it follows from (1.33), (1.35), (2.2)[1] that

$$\pi_{22}\pi_{31} + \pi_{23}(\pi_{21} - \pi_{12}) - 2\pi_{21}\pi_{32} - \frac{1}{2}\frac{\alpha_{13}}{\alpha_{12}}u_1 = 0.$$

and for (9.2) to give non-zero values of π_{12} and π_{13} in accordance with Lemmas 8.1 and 8.2 we must have

$$\pi_{22}\pi_{33} = \pi_{23}\pi_{32} = \pi_{23}^2, \quad \text{by (8.26)}. \tag{9.3}$$

From (8.3) and (8.26)

$$\pi_{13}\pi_{21} - \pi_{31}\pi_{12} = 0.$$

These conditions are combined in the form

$$\frac{\pi_{13}}{\pi_{12}} = \frac{\pi_{33}}{\pi_{23}} = \frac{\pi_{23}}{\pi_{22}} = \frac{\pi_{31}}{\pi_{21}}. \tag{9.4}$$

From (8.6), (8.24) and (9.1)

$$x_3 = -\pi_{33} \operatorname{div} \boldsymbol{e}_3, \qquad y_2 = \pi_{22} \operatorname{div} \boldsymbol{e}_2. \tag{9.5}$$

Since $\frac{\alpha_{32}}{\alpha_{31}}$ is does not vanish for distinct proper numbers, the condition (8.6) with (1.22), (8.24) and (8.26) gives $x_1 = y_1$. Using $\frac{\delta}{\delta e_2}\left(\frac{\pi_{21}}{\pi_{22}}\right) = 0$ with (1.35), $(2.2)^1$ and (3.8), we then get

$$x_1 = y_1 = \pi_{21}(2\pi_{13} - \pi_{31}) - \pi_{11}\pi_{23} \tag{9.6}$$

and it follows from (3.6), (9.4) and (9.6) that

$$\frac{\delta}{\delta e_2} \operatorname{div} \boldsymbol{e}_3 = \frac{\delta}{\delta e_3} \operatorname{div} \boldsymbol{e}_2 = \pi_{13}\pi_{21} - \pi_{11}\pi_{23}. \tag{9.7}$$

We note that the conditions (1.24) to (1.27) may be written

$$\begin{aligned}\frac{\delta}{\delta e_3}\pi_{21} &= (\pi_{21}^2 - \pi_{11}\pi_{22}) - \frac{\alpha_{12}}{\alpha_{32}}(\pi_{32}\pi_{23} - \pi_{22}\pi_{33}) - \frac{\alpha_{23}}{\alpha_{21}}\lambda_3\pi_{33} \\ &\quad + \frac{3}{4}\lambda_2\lambda_3 + \frac{1}{4}\left[\frac{\alpha_{23}}{\alpha_{21}} - \frac{\alpha_{12}}{\alpha_{32}}\right]\lambda_3\lambda_1 - \frac{1}{4}\frac{\alpha_{13}\alpha_{21}}{\alpha_{32}\alpha_{23}}\lambda_1\lambda_2, \quad \text{etc.},\end{aligned} \tag{9.8}$$

and

$$\begin{aligned}\frac{\delta}{\delta e_2}\pi_{31} &= (\pi_{11}\pi_{33} - \pi_{31}^2) - \frac{\alpha_{13}}{\alpha_{23}}(\pi_{22}\pi_{33} - \pi_{23}\pi_{32}) - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\lambda_2 \\ &\quad + \frac{3}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3 + \frac{1}{4}\frac{\alpha_{13}}{\alpha_{23}}\lambda_3\lambda_1 + \frac{1}{4}\left[1 - \frac{\alpha_{13}\alpha_{31}}{\alpha_{32}\alpha_{23}}\right]\lambda_1\lambda_2, \quad \text{etc.}\end{aligned} \tag{9.9}$$

By Lemma 8.6 all the curvatures bear constant values on the vector-lines of $\boldsymbol{e}_1$, and we have in particular $\frac{\delta}{\delta e_1}\pi_{32} = 0$, $\frac{\delta}{\delta e_1}\pi_{23} = 0$. Using (8.24), (9.3), and (1.22), we obtain from (9.8) and (9.9)

$$\pi_{33}\pi_{11} - \pi_{13}\pi_{31} = -\frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3, \tag{9.10}$$

$$\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = \frac{1}{4}\lambda_2\lambda_3. \tag{9.11}$$

Again by (9.3), (9.8) and (9.9)

$$\frac{\delta}{\delta e_3}\operatorname{div} e_3 = \frac{\delta}{\delta e_3}(\pi_{12} - \pi_{21}) = (\pi_{22}\pi_{11} - \pi_{12}^2) - \frac{\alpha_{21}}{\alpha_{31}}(\pi_{33}\pi_{11} - \pi_{31}\pi_{13}) - \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}\lambda_3$$
$$+\frac{1}{4}\left[1 - \frac{\alpha_{21}\alpha_{12}}{\alpha_{13}\alpha_{31}}\right]\lambda_2\lambda_3 - (\pi_{21}^2 - \pi_{11}\pi_{22}) + \frac{\alpha_{23}}{\alpha_{21}}\pi_{33}\lambda_3 - \frac{3}{4}\lambda_2\lambda_3,$$

and by (9.10) and (9.11) we obtain

$$\frac{\delta}{\delta e_3}\operatorname{div} e_3 = -(\operatorname{div} e_3)^2. \tag{9.12}$$

Similarly we obtain

$$\frac{\delta}{\delta e_2}\operatorname{div} e_2 = -(\operatorname{div} e_2)^2. \tag{9.13}$$

Let e_3^* be the unit normal to the surface π_{11} = constant determined in Lemma 8.6; then $e_2^* = e_3^* \times e_1$ is the unit vector perpendicular to e_1 in the tangent plane of the surface. Then by (8.4) and (8.26)

$$e_3^* = \frac{-\operatorname{div} e_2 e_2 - \operatorname{div} e_3 e_3}{D}$$

and

$$e_2^* = \frac{-\operatorname{div} e_3 e_2 + \operatorname{div} e_2 e_3}{D}. \tag{9.14}$$

By Lemma 8.6 all the curvatures are constant on the surface π_{11} = constant. It follows that

$$\frac{\delta}{\delta e_2^*}\operatorname{div} e_2 = 0 \quad \text{and} \quad \frac{\delta}{\delta e_2^*}\operatorname{div} e_3 = 0. \tag{9.15}$$

From $(9.14)^2$ and (9.15)

$$-\operatorname{div} e_3 \frac{\delta}{\delta e_2}\operatorname{div} e_2 + \operatorname{div} e_2 \frac{\delta}{\delta e_3}\operatorname{div} e_2 = 0 \tag{9.16}$$

and

$$-\operatorname{div} e_3 \frac{\delta}{\delta e_2}\operatorname{div} e_3 + \operatorname{div} e_2 \frac{\delta}{\delta e_3}\operatorname{div} e_3 = 0. \tag{9.17}$$

Substituting for $\frac{\delta}{\delta e_3}\operatorname{div} e_3$ and $\frac{\delta}{\delta e_2}\operatorname{div} e_2$ from (9.12) and (9.13) and noting from Lemma 8.7 that neither $\operatorname{div} e_2$ nor $\operatorname{div} e_3$ can vanish, we obtain

$$\frac{\delta}{\delta e_3}\operatorname{div} e_2 = \frac{\delta}{\delta e_2}\operatorname{div} e_3 = -\operatorname{div} e_2 \operatorname{div} e_3. \tag{9.18}$$

From (9.7) and (9.18)

$$-\operatorname{div} e_2 \operatorname{div} e_3 = \pi_{13}\pi_{21} - \pi_{11}\pi_{23}. \tag{9.19}$$

By (8.27), (8.28), and (9.19), using (1.22) we obtain

$$\frac{\alpha_{13}}{\alpha_{12}}\pi_{11}\pi_{33}(\pi_{13}\pi_{21} - \pi_{11}\pi_{23}) - \pi_{12}\pi_{31}\lambda_2\lambda_3 = 0. \tag{9.20}$$

We write

$$\pi_{13}\pi_{21} = \pi_{12}\pi_{31} \stackrel{\text{def}}{=} x, \qquad \pi_{23} \stackrel{\text{def}}{=} y, \tag{9.21}$$

and (9.20) becomes

$$\left[\pi_{11}\pi_{33} - \frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3\right] x - \pi_{11}^2\pi_{33}\, y = 0. \tag{9.22}$$

From (9.4) $\pi_{13}\pi_{31} = \pi_{33}\left(\frac{\pi_{21}\pi_{13}}{\pi_{23}}\right)$ so that

$$\pi_{13}\pi_{31} = \pi_{33}\frac{x}{y},$$

and we obtain from (9.10)

$$-\pi_{33}x + \left[\pi_{11}\pi_{33} + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3\right] y = 0. \tag{9.23}$$

For (9.22) and (9.23) to give non-vanishing values for x and y one must have

$$\pi_{11}^2\pi_{33}^2 - \left[\pi_{11}\pi_{33} - \frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3\right]\left[\pi_{11}\pi_{33} + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3\right] = 0$$

which reduces to

$$\lambda_2\lambda_3\left(3\pi_{11}\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3\right) = 0. \tag{9.24}$$

It follows from (9.24) that

$$\lambda_2\lambda_3 = 0 \tag{9.25}$$

or

$$3\pi_{11}\pi_{33} + \frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3 = 0. \tag{9.26}$$

If (9.25) holds, the result follows from Lemma 8.8.

We are left with the possibility (9.26). Using the expressions (2.7) for λ_2 and λ_3, and eliminating π_{33} in favor of π_{22} by (8.24), and using (1.22) and (8.24), we reduce (9.26) to

$$-\pi_{11}\pi_{22} + \frac{\alpha_{31}}{\alpha_{32}}\pi_{11}^2 + \frac{\alpha_{32}}{\alpha_{31}}\pi_{22}^2 = 0. \tag{9.27}$$

Our aim now is to obtain a further expression involving only the abnormalities and the ratios of the proper numbers. We achieve this by obtaining an expression for $\frac{\delta}{\delta e_2}\pi_{13}$ in terms of the abnormalities and then eliminating $\frac{\delta}{\delta e_2}\pi_{13}$ between this expression and (9.8). The curvatures are eliminated in favor of the abnormalities by (8.2), (9.10) and (9.11).

We begin by taking the gradient of (9.19) with respect to e_2. Some subsidiary expressions are needed.

From (9.13) and (9.18) we have

$$\frac{\delta}{\delta e_2}(\operatorname{div} e_2 \operatorname{div} e_3) = -2(\operatorname{div} e_2)^2 \operatorname{div} e_3, \tag{9.28}$$

and by (1.35), (3.9), $(9.2)^1$ and $(9.5)^1$

$$\frac{\delta}{\delta e_2}\pi_{23} = -x_3 + \pi_{23}\pi_{13} - \pi_{21}\pi_{33} = 2\pi_{33}\operatorname{div} e_3. \tag{9.29}$$

By (8.1) and Lemma 8.3, $\frac{\delta}{\delta e_2}\frac{\pi_{21}}{\pi_{22}} = 0$. Accordingly, by (2.2), one has

$$\frac{\delta}{\delta e_2}\pi_{21} = -2\pi_{21}\operatorname{div} e_2. \tag{9.30}$$

Equation (9.30) also follows from (3.8) and (9.6).

By (8.4), (9.28), (9.29) and (9.30) we obtain for the e_2-gradient of (9.19)

$$\begin{aligned} 2(\operatorname{div} e_2)^2 \operatorname{div} e_3 &= \left(\frac{\delta}{\delta e_2}\pi_{13}\right)\pi_{21} \\ &\quad - 2(\pi_{13}\pi_{21} - \pi_{11}\pi_{23})\operatorname{div} e_2 - 2\pi_{11}\pi_{33}\operatorname{div} e_3, \end{aligned}$$

which by (9.19) reduces to

$$\pi_{21}\frac{\delta}{\delta e_2}(\pi_{13}) - 2\pi_{11}\pi_{33}\operatorname{div} e_3 = 0. \tag{9.31}$$

By $(8.2)^2$ and (9.31), eliminating π_{12} in favor of π_{21} and then cancelling π_{21} which by Lemma 8.2 does not vanish, we get

$$\pi_{22}\left(\frac{\delta}{\delta e_2}\pi_{13} + 2\pi_{11}\pi_{33}\right) + 2\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}^2\pi_{33} = 0, \tag{9.32}$$

which is the required expression for $\frac{\delta}{\delta e_2}\pi_{13}$.

By (9.8) and (9.11) for $\lambda_1 = 0$, we have

$$\frac{\delta}{\delta e_2}\pi_{13} = \pi_{13}^2 - \pi_{33}\pi_{11} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\lambda_2 + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3. \tag{9.33}$$

Also by (8.2)

$$\pi_{13}^2 = -\frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}}\pi_{13}\pi_{31} = -\frac{\alpha_{21}}{\alpha_{23}}\frac{\pi_{11}}{\pi_{33}}\left[\pi_{33}\pi_{11} + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\lambda_2\lambda_3\right], \tag{9.34}$$

by (9.10). Eliminating π_{13}^2 from (9.33) by means of (9.34) and then substituting for $\frac{\delta}{\delta e_2}\pi_{13}$ in (9.32), we get

$$\begin{aligned}\pi_{22}&\left[-\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2 + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\left(1 - \frac{\pi_{11}}{\pi_{33}}\frac{\alpha_{21}}{\alpha_{23}}\right)\lambda_2\lambda_3 + \pi_{33}\pi_{11} - \frac{\alpha_{12}}{\alpha_{13}}\pi_{22}\lambda_2\right]\\ &+\frac{2\alpha_{31}}{\alpha_{32}}\pi_{11}^2\pi_{33} = 0.\end{aligned} \tag{9.35}$$

By (8.24) and (1.22)

$$2\frac{\alpha_{31}}{\alpha_{32}}\pi_{11}^2\pi_{33} = -2\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}^2\pi_{22} = 2\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2\pi_{22},$$

so (9.35) reduces to

$$\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2 + \frac{1}{4}\frac{\alpha_{12}}{\alpha_{13}}\left(1 - \frac{\pi_{11}}{\pi_{33}}\frac{\alpha_{21}}{\alpha_{23}}\right)\lambda_2\lambda_3 + \pi_{33}\pi_{11} - \frac{\alpha_{12}}{\alpha_{13}}\lambda_2\pi_{22} = 0. \tag{9.36}$$

Substituting for $\lambda_2\lambda_3$ from (9.26), substituting for λ_2 from (2.7), and eliminating π_{33} in favor of π_{22} by (8.24), we reduce (9.36) to

$$\frac{7}{4}\frac{\alpha_{21}}{\alpha_{23}}\pi_{11}^2 - \frac{5}{4}\frac{\alpha_{12}}{\alpha_{13}}\pi_{11}\pi_{22} - \frac{\alpha_{32}\alpha_{12}}{\alpha_{31}\alpha_{13}}\pi_{22}^2 = 0. \tag{9.37}$$

Eliminating π_{22}^2 between (9.27) and (9.37) and using (1.22), since π_{11} does not vanish, we obtain

$$\pi_{11} = -3\frac{\alpha_{32}}{\alpha_{31}}\pi_{22}. \tag{9.38}$$

Substituting (9.38) into (9.27), we get

$$13\frac{\alpha_{32}}{\alpha_{31}}\pi_{22}^2 = 0. \tag{9.39}$$

For distinct proper numbers the ratio $\frac{\alpha_{32}}{\alpha_{31}}$ does not vanish. Thus π_{22} is zero, which is a contradiction. This proves Main Theorem 2.

Concluding Remarks

No simple new conditions are obtainable by applying the commutation formulae (1.11) to the curvatures and abnormalities. The evaluation of expressions such as

$$\left(\frac{\delta^3}{\delta e_3\,\delta e_2\,\delta e_1} - \frac{\delta^3}{\delta e_2\,\delta e_3\,\delta e_1}\right)\pi_{23}$$

gives identities.

It was hoped that another set of integrals of the same order as (5.7), (6.9) and (6.10) could be obtained by taking the directional derivative of the symmetrical condition (5.2). The procedure is as follows.

Taking the gradient of (5.2) with respect to $\boldsymbol{e}_1$, we obtain the second gradients $\frac{\delta^2 u_3}{\delta e_1^2}, \frac{\delta^2 u_1}{\delta e_1\,\delta e_2}, \frac{\delta^2 u_2}{\delta e_1\,\delta e_3}$. The two mixed gradients are reduced by the commutation formula (1.11). The gradient $\frac{\delta^2 u_3}{\delta e_1^2}$ can be expressed in terms of $\frac{\delta^2 x_2}{\delta e_1\,\delta e_2}$ by (4.10), and hence, using (1.11), in terms of $\frac{\delta^3}{\delta e_2\,\delta e_3\,\delta e_1}(\pi_{23} - \pi_{32})$ by (4.3). This term can eventually be reduced. Alternatively, $\frac{\delta^2 u_3}{\delta e_1^2}$ can be expressed in terms of $\frac{\delta^2 y_3}{\delta e_1\,\delta e_3}$ by (4.9), and a similar, dual procedure adopted. Before realizing that this opening of two paths strongly indicated that the calculations would lead to an identity, we carried out the procedure completely and indeed obtained an identity. The reduction requires the expressions (6.9) and (6.10) and gives a verification of their correctness.

While Main Theorem I.1 guarantees one homogeneous polynomial relation among the abnormalities, the curvatures and the two independent ratios of proper numbers, we cannot be sure that the ten integrals given by (5.2), (5.7), (6.9) and (6.10) are completely independent. With the large number of terms involved we cannot be sure that common factors representing new solutions will not arise in later stages of the elimination process. However, if these integrals are completely independent and if we can generate six further completely independent integrals, then we would have sufficient conditions to show that the ratios of the abnormalities are constant. It would then follow from Main Theorem I.2 that no new solutions are possible.

We have established that no new integrals can be obtained by taking the gradients of the symmetrical condition (5.2). It is an impossibly tedious task to derive completely the conditions determined by taking the gradients of (5.7), (6.9) and (6.10). We may note that in the expressions obtained as the $\boldsymbol{e}_3$-gradients of (6.9) and (6.10)* the highest terms in u_a come from

$$u_3\left[\frac{\alpha_{12}\alpha_{32}}{\alpha_{13}\alpha_{31}}\,\frac{(y_3 - 2x_3)\,u_3}{\pi_{13}} + \frac{(x_3 - 2y_3)\,u_1}{\pi_{23}}\right], \quad \text{etc.}$$

---

* We use (4.9) and (4.10) to give $\frac{\delta y_3}{\delta e_3}$ and $\frac{\delta x_3}{\delta e_3}$ in terms of $\frac{\delta u_3}{\delta e_1}$ and $\frac{\delta u_1}{\delta e_2}$ and then eliminate the latter gradients using the original expressions (6.9) and (6.10).

and

$$\frac{\alpha_{13}}{\alpha_{12}} u_1 \left[\frac{\alpha_{12}\alpha_{32}}{\alpha_{13}\alpha_{31}} \frac{(y_3 - 2x_3) u_3}{\pi_{13}} + \frac{(x_3 - 2y_3) u_1}{\pi_{23}}\right], \quad \text{etc.},$$

respectively. The highest terms in u_a in the gradients of (6.9) with respect to $\boldsymbol{e}_1$ and the gradient of (6.10) with respect to $\boldsymbol{e}_2$ come from

$$-\frac{2\alpha_{12}}{\alpha_{13}} \frac{(y_3 - x_3)(y_3 - 2x_3) u_3}{\pi_{13}}, \quad \text{etc.}$$

and

$$+\frac{2(y_3 - x_3)(x_3 - 2y_3) u_1}{\pi_{23}}, \quad \text{etc.},$$

respectively. The pairs of leading terms show common factors. The complete expressions may possess corresponding common factors, leaving at most six conditions which in turn may not be independent.

It seems possible that the verification or otherwise of the complete independence of the ten integrals (5.2), (5.7), (6.9), and (6.10), the determination of the conditions given by the gradients of (5.7), (6.9) and (6.10) and finally the performance of the eliminations, could be accomplished using the computer symbolic mathematics systems (1979 [1]). For the benefit of a reader who might wish to undertake this we tabulate the equations giving the various gradients required for deriving the conditions from (5.7), (6.9) and (6.10).

Gradient	Equation
$\frac{\delta}{\delta e_3}\pi_{21}, \frac{\delta}{\delta e_3}\pi_{12}$, etc.	(1.24), (1.26)
$\frac{\delta}{\delta e_3}\pi_{23}, \frac{\delta}{\delta e_3}\pi_{13}$, etc.	(1.32), (1.33)
$\frac{\delta}{\delta e_3}\pi_{32}, \frac{\delta}{\delta e_3}\pi_{31}$, etc.	(3.8), (3.9)
$\frac{\delta}{\delta e_3}\pi_{11}, \frac{\delta}{\delta e_3}\pi_{22}, \frac{\delta}{\delta e_3}\pi_{33}$, etc.	(1.29), (1.30), (2.2)
$\frac{\delta u_1}{\delta e_1}, \frac{\delta u_1}{\delta e_3}$, etc.	(3.15), (3.16)
$\frac{\delta x_1}{\delta e_1}, \frac{\delta x_1}{\delta e_2}, \frac{\delta x_1}{\delta e_3}$, etc.	(4.10), (4.1), (4.3)
$\frac{\delta y_1}{\delta e_1}, \frac{\delta y_1}{\delta e_2}, \frac{\delta y_1}{\delta e_3}$, etc.	(4.9), (4.2), (4.4)
$\frac{\delta u_1}{\delta e_2}$, etc.	(6.9), (6.10)

Note added in proof. It seems that the statement "We give ten polynomial integrals effectively involving sixteen arguments. These are the ten variables indicated above and ..." following the statement of Main Theorem 2 in the Introduction, needs clarification.

In (I.2) there are two independent ratios of proper numbers. Also, the relation (I.2) and all other conditions obtained after eliminating u_1, u_2, u_3, $\frac{\delta u_1}{\delta e_1}, \frac{\delta u_2}{\delta e_3}, \frac{\delta u_3}{\delta e_1}$, are homogeneous polynomials in the nine abnormalities and curvatures. The abnormalities do not vanish. We may thus divide throughout by one of the abnormalities, π_{11} say, to the appropriate power, and see the conditions as relations among the eight ratios $\frac{\pi_{22}}{\pi_{11}}, \frac{\pi_{33}}{\pi_{11}}, \frac{\pi_{12}}{\pi_{11}}, \ldots$. These ratios of the proper numbers comprise the ten variables referred to.

Acknowledgments. The author expresses his appreciation to the U. S. National Science Foundation, which supported this study by a grant. He is indebted to Mrs. Van Hook for her care and patience in typing this work. He acknowledges the help of Professor W.-L. Yin. Through his long experience with this problem he has repeatedly saved the writer from emotionally inspired errors in logic. He expresses his thanks to Professor C.-C. Wang for all the time he has devoted to this paper and for his many constructive suggestions.

References

1954 [1] Ericksen, J. L., "Deformations possible in every isotropic incompressible perfectly elastic body", *Z. angew. Math. Phys.* **5**, 466–486.

1965 [1] Singh, M., & A. C. Pipkin, "Note on Ericksen's problem", *Z. angew. Math. Phys.* **16**, 706–709.

1966 [1] Fosdick, R. L., "Remarks on compatibility", *Modern Developments in the Mechanics of Continua.* Academic Press, New York, 109–127.

1966 [2] Klingbeil, W. W., & R. T. Shield, "On a class of solutions in plane finite elasticity", *Z. angew. Math. Phys.* **17**, 489–511.

1969 [1] Fosdick, R. L., & K. W. Schuler, "On Ericksen's problem for plane deformations with uniform transverse stretch", *Int. J. Eng. Science* **7**, 217–233.

1970 [1] Marris, A. W., & J. F. Shiau, "Universal deformations in isotropic incompressible hyperelastic materials when the deformation tensor has equal proper values", *Arch. Rational Mech. Anal* **36**, 135–160.

1971 [1] Fosdick, R. L., "Statically possible radially symmetric deformations in isotropic, incompressible elastic solids", *Journal of Applied Mathematics and Physics (ZAMP)* **22**, 590–607.

1972 [1] Kafadar, C. B., "On Ericksen's problem", *Arch. Rational Mech. Anal.*, **47**, 15–27.

1975 [1] Marris, A. W., "Universal deformations in incompressible isotropic elastic materials", *Journal of Elasticity*, **5**, 111–128.

1979 [1] Stoutemyer, D. R., "Symbolic computation comes of age", *SIAM News*, **12**, 6, 1–2.

School of Engineering Science
and Mechanics
Georgia Institute of Technology
Atlanta

(Received September 21, 1981)